THE NATURE OF GRAPHS

CHAPTER OBJECTIVES

- **Graph functions, relations, inverses, and inequalities.** *(Lessons 3-1, 3-3, 3-4, 3-7)*
- **Analyze families of graphs.** *(Lesson 3-2)*
- **Investigate symmetry, continuity, end behavior, and transformations of graphs.** *(Lessons 3-1, 3-2, 3-5)*
- **Find asymptotes and extrema of functions.** *(Lessons 3-6, 3-7)*
- **Solve problems involving direct, inverse, and joint variation.** *(Lesson 3-8)*

3-1 Symmetry and Coordinate Graphs

OBJECTIVES
- Use algebraic tests to determine if the graph of a relation is symmetrical.
- Classify functions as even or odd.

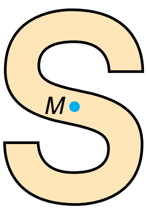
PHARMACOLOGY Designing a drug to treat a disease requires an understanding of the molecular structures of the substances involved in the disease process. The substances are isolated in crystalline form, and X rays are passed through the symmetrically-arranged atoms of the crystals. The existence of symmetry in crystals causes the X rays to be diffracted in regular patterns. These symmetrical patterns are used to determine and visualize the molecular structure of the substance. *A related problem is solved in Example 4.*

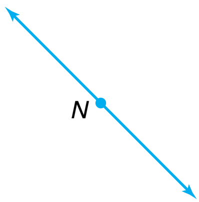
Like crystals, graphs of certain functions display special types of symmetry. For some functions with symmetrical graphs, knowledge of symmetry can often help you sketch and analyze the graphs. One type of symmetry a graph may have is **point symmetry.**

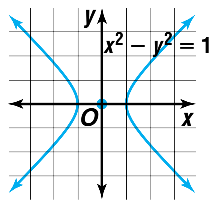
Point Symmetry Two distinct points P and P' are symmetric with respect to point M if and only if M is the midpoint of $\overline{PP'}$. Point M is symmetric with respect to itself.

When the definition of point symmetry is extended to a set of points, such as the graph of a function, then each point P in the set must have an **image point** P' that is also in the set. A figure that is symmetric with respect to a given point can be rotated 180° about that point and appear unchanged. Each of the figures below has point symmetry with respect to the labeled point.

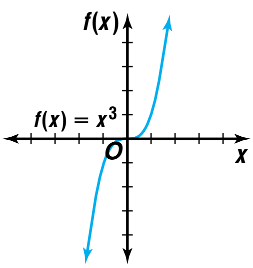
Symmetry with respect to a given point M can be expressed as symmetry about point M.





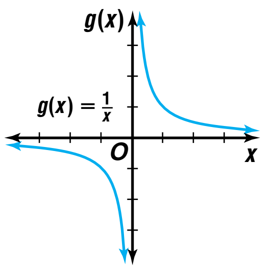


The origin is a common point of symmetry. Observe that the graphs of $f(x) = x^3$ and $g(x) = \frac{1}{x}$ exhibit symmetry with respect to the origin. Look for patterns in the table of function values beside each graph.



$f(x) = x^3$			
x	$f(x)$	$f(-x)$	$-f(x)$
1	1	-1	-1
2	8	-8	-8
3	27	-27	-27
4	64	-64	-64

Note that $f(-x) = -f(x)$.



$g(x) = \frac{1}{x}$			
x	$g(x)$	$g(-x)$	$-g(x)$
1	1	-1	-1
2	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
3	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$

Note that $g(-x) = -g(x)$.

The values in the tables suggest that $f(-x) = -f(x)$ whenever the graph of a function is symmetric with respect to the origin.

Symmetry with Respect to the Origin

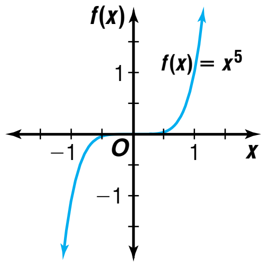
The graph of a relation S is symmetric with respect to the origin if and only if $(a, b) \in S$ implies that $(-a, -b) \in S$. A function has a graph that is symmetric with respect to the origin if and only if $f(-x) = -f(x)$ for all x in the domain of f.

$(a, b) \in S$ means the ordered pair (a, b) belongs to the solution set S.

Example 1 demonstrates how to algebraically test for symmetry about the origin.

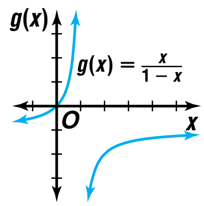
Example 1 **Determine whether each graph is symmetric with respect to the origin.**

a. $f(x) = x^5$



The graph of $f(x) = x^5$ appears to be symmetric with respect to the origin.

b. $g(x) = \dfrac{x}{1 - x}$



The graph of $g(x) = \dfrac{x}{1 - x}$ does not appear to be symmetric with respect to the origin.

We can verify these conjectures algebraically by following these two steps.

1. Find $f(-x)$ and $-f(x)$.
2. If $f(-x) = -f(x)$, the graph has point symmetry.

a. $f(x) = x^5$

Find $f(-x)$.

$f(-x) = (-x)^5$ *Replace x with $-x$.*

$f(-x) = -x^5$ *$(-x)^5 = (-1)^5x^5 = -1x^5$ or $-x^5$*

Find $-f(x)$.

$-f(x) = -x^5$ *Determine the opposite of the function.*

The graph of $f(x) = x^5$ is symmetric with respect to the origin because $f(-x) = -f(x)$.

b. $g(x) = \dfrac{x}{1 - x}$

Find $g(-x)$.

$$g(-x) = \frac{-x}{1 - (-x)} = \frac{-x}{1 + x}$$ *Replace x with $-x$.*

Find $-g(x)$.

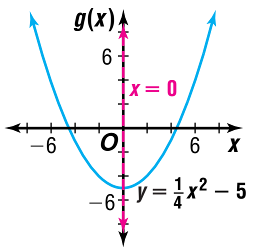
$$-g(x) = -\frac{x}{1 - x} = \frac{-x}{1 - x}$$ *Determine the opposite of the function.*

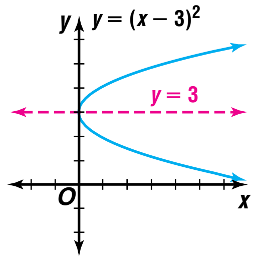
The graph of $g(x) = \dfrac{x}{1 - x}$ is not symmetric with respect to the origin because $g(-x) \neq -g(x)$.

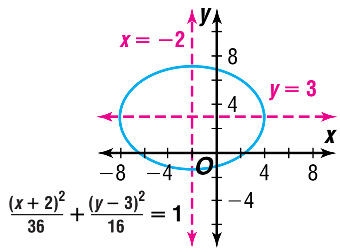
Another type of symmetry is **line symmetry.**

Line Symmetry	Two distinct points P and P' are symmetric with respect to a line ℓ if and only if ℓ is the perpendicular bisector of $\overline{PP'}$. A point P is symmetric to itself with respect to line ℓ if and only if P is on ℓ.

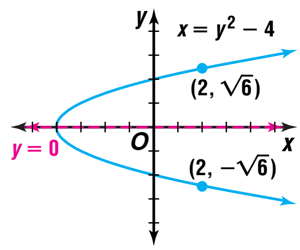
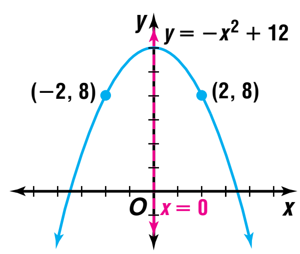
Each graph below has line symmetry. The equation of each line of symmetry is given. Graphs that have line symmetry can be folded along the line of symmetry so that the two halves match exactly. Some graphs, such as the graph of an ellipse, have more than one line of symmetry.



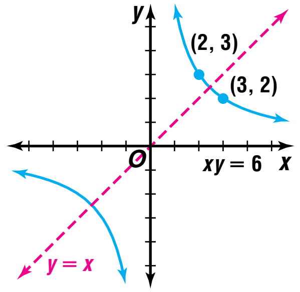
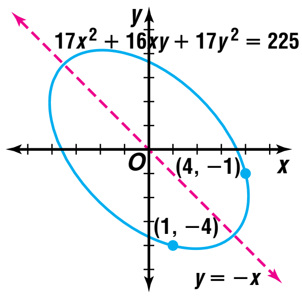




Some common lines of symmetry are the x-axis, the y-axis, the line $y = x$, and the line $y = -x$. The following table shows how the coordinates of symmetric points are related for each of these lines of symmetry. Set notation is often used to define the conditions for symmetry.

Symmetry with Respect to the:	Definition and Test	Example
x-axis	$(a, -b) \in S$ if and only if $(a, b) \in S$. *Example:* $(2, \sqrt{6})$ and $(2, -\sqrt{6})$ are on the graph. *Test:* Substituting (a, b) and $(a, -b)$ into the equation produces equivalent equations.	
y-axis	$(-a, b) \in S$ if and only if $(a, b) \in S$. *Example:* $(2, 8)$ and $(-2, 8)$ are on the graph. *Test:* Substituting (a, b) and $(-a, b)$ into the equation produces equivalent equations.	

(continued on the next page)

Symmetry with Respect to the Line:	Definition and Test	Example
$y = x$	$(b, a) \in S$ if and only if $(a, b) \in S$. *Example:* $(2, 3)$ and $(3, 2)$ are on the graph. *Test:* Substituting (a, b) and (b, a) into the equation produces equivalent equations.	
$y = -x$	$(-b, -a) \in S$ if and only if $(a, b) \in S$. *Example:* $(4, -1)$ and $(1, -4)$ are on the graph. *Test:* Substituting (a, b) and $(-b, -a)$ into the equation produces equivalent equations.	

You can determine whether the graph of an equation has line symmetry without actually graphing the equation.

Example 2 **Determine whether the graph of $xy = -2$ is symmetric with respect to the x-axis, y-axis, the line $y = x$, the line $y = -x$, or none of these.**

Substituting (a, b) into the equation yields $ab = -2$. Check to see if each test produces an equation equivalent to $ab = -2$.

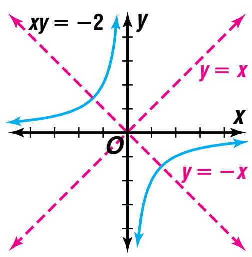
x-axis
$$\begin{aligned} a(-b) &= -2 && \textit{Substitute } (a, -b) \textit{ into the equation.}\\ -ab &= -2 && \textit{Simplify.}\\ ab &= 2 && \textit{Not equivalent to } ab = -2 \end{aligned}$$

y-axis
$$\begin{aligned} (-a)b &= -2 && \textit{Substitute } (-a, b) \textit{ into the equation.}\\ -ab &= -2 && \textit{Simplify.}\\ ab &= 2 && \textit{Not equivalent to } ab = -2 \end{aligned}$$

$y = x$
$$\begin{aligned} (b)(a) &= -2 && \textit{Substitute } (b, a) \textit{ into the equation.}\\ ab &= -2 && \textit{Equivalent to } ab = -2 \end{aligned}$$

$y = -x$
$$\begin{aligned} (-b)(-a) &= -2 && \textit{Substitute } (-b, -a) \textit{ into the equation.}\\ ab &= -2 && \textit{Equivalent to } ab = -2 \end{aligned}$$

Therefore, the graph of $xy = -2$ is symmetric with respect to the line $y = x$ and the line $y = -x$. A sketch of the graph verifies the algebraic tests.



You can use information about symmetry to draw the graph of a relation.

Example 3 **Determine whether the graph of $|y| = 2 - |2x|$ is symmetric with respect to the x-axis, the y-axis, both, or neither. Use the information about the equation's symmetry to graph the relation.**

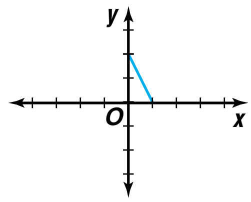
Substituting (a, b) into the equation yields $|b| = 2 - |2a|$. Check to see if each test produces an equation equivalent to $|b| = 2 - |2a|$.

x-axis	$\lvert -b\rvert = 2 - \lvert 2a\rvert$	*Substitute $(a, -b)$ into the equation.*
	$\lvert b\rvert = 2 - \lvert 2a\rvert$	*Equivalent to $\lvert b\rvert = 2 - \lvert 2a\rvert$ since $\lvert -b\rvert = \lvert b\rvert$.*
y-axis	$\lvert b\rvert = 2 - \lvert -2a\rvert$	*Substitute $(-a, b)$ into the equation.*
	$\lvert b\rvert = 2 - \lvert 2a\rvert$	*Equivalent to $\lvert b\rvert = 2 - \lvert 2a\rvert$, since $\lvert -2a\rvert = \lvert 2a\rvert$.*

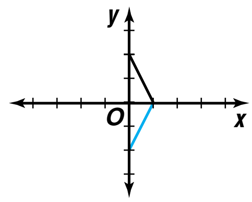
Therefore, the graph of $|y| = 2 - |2x|$ is symmetric with respect to both the x-axis and the y-axis.

To graph the relation, let us first consider ordered pairs where $x \geq 0$ and $y \geq 0$. The relation $|y| = 2 - |2x|$ contains the same points as $y = 2 - 2x$ in the first quadrant.

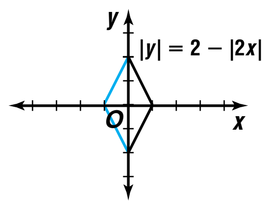
Therefore, in the first quadrant, the graph of $|y| = 2 - |2x|$ is the same as the graph of $y = 2 - 2x$.



Since the graph is symmetric with respect to the x-axis, every point in the first quadrant has a corresponding point in the fourth quadrant.

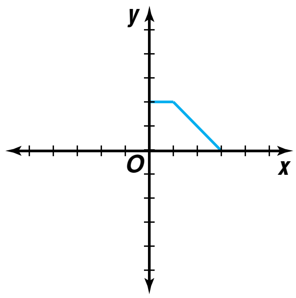


Since the graph is symmetric with respect to the y-axis, every point in the first and fourth quadrants has a corresponding point on the other side of the y-axis.



Example 4 Real World Application

CRYSTALLOGRAPHY **A crystallographer can model a cross-section of a crystal with mathematical equations. After sketching the outline on a graph, she notes that the crystal has both x-axis and y-axis symmetry. She uses the piecewise function $y = \begin{cases} 2 \text{ if } 0 \leq x \leq 1 \\ 3 - x \text{ if } 1 \leq x \leq 3 \end{cases}$ to model the first quadrant portion of the cross-section. Write piecewise equations for the remaining sides.**



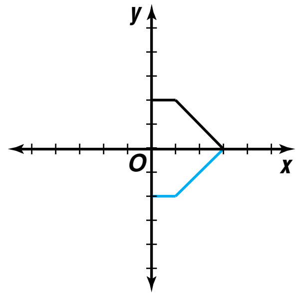
> **Look Back**
> Refer to Lesson 1-7 for more about piecewise functions.

Since the graph has x-axis symmetry, substitute $(x, -y)$ into the original equation to produce the equation for the fourth quadrant portion.

$$y = \begin{cases} 2 \text{ if } 0 \leq x \leq 1 \\ 3 - x \text{ if } 1 \leq x \leq 3 \end{cases}$$

$$-y = \begin{cases} 2 \text{ if } 0 \leq x \leq 1 \\ 3 - x \text{ if } 1 \leq x \leq 3 \end{cases}$$ *Substitute $(x, -y)$ into the equation.*

$$y = \begin{cases} -2 \text{ if } 0 \leq x \leq 1 \\ 3 + x \text{ if } 1 \leq x \leq 3 \end{cases}$$ *Solve for y.*



The equation $y = \begin{cases} -2 \text{ if } 0 \leq x \leq 1 \\ x - 3 \text{ if } 1 \leq x \leq 3 \end{cases}$ models the fourth quadrant portion of the cross section.

Since the graph has y-axis symmetry, substitute $(-x, y)$ into the first and fourth quadrant equations to produce the equations for the second and third quadrants.

$$y = \begin{cases} 2 \text{ if } 0 \leq x \leq 1 \\ 3 - x \text{ if } 1 \leq x \leq 3 \end{cases}$$ *Start with the first quadrant equation.*

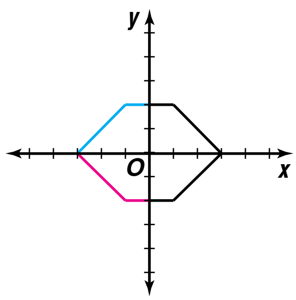
$$y = \begin{cases} 2 \text{ if } 0 \leq -x \leq 1 \\ 3 - (-x) \text{ if } 1 \leq -x \leq 3 \end{cases}$$ *Substitute $(-x, y)$ into the equation.*

$$y = \begin{cases} 2 \text{ if } 0 \geq x \geq -1 \\ 3 + x \text{ if } -1 \geq x \geq -3 \end{cases}$$ *This is the second quadrant equation.*

$$y = \begin{cases} -2 \text{ if } 0 \leq x \leq 1 \\ x - 3 \text{ if } 1 \leq x \leq 3 \end{cases}$$ *Start with the fourth quadrant equation.*

$$y = \begin{cases} -2 \text{ if } 0 \leq -x \leq 1 \\ -(-x) - 3 \text{ if } 1 \leq -x \leq 3 \end{cases}$$ *Substitute $(-x, y)$ into the equation.*

$$y = \begin{cases} -2 \text{ if } 0 \geq x \geq -1 \\ x - 3 \text{ if } -1 \geq x \geq -3 \end{cases}$$ *This is the third quadrant equation.*

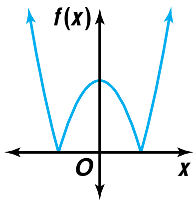
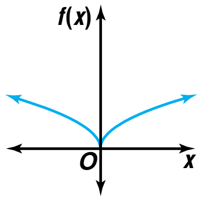
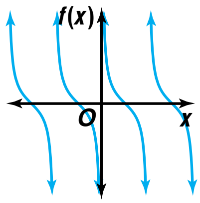
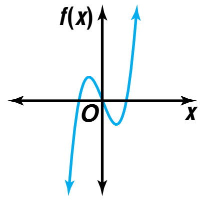


The equations $y = \begin{cases} 2 \text{ if } -1 \leq x \leq 0 \\ 3 + x \text{ if } -3 \leq x \leq -1 \end{cases}$ and $y = \begin{cases} -2 \text{ if } -1 \leq x \leq 0 \\ x - 3 \text{ if } -3 \leq x \leq -1 \end{cases}$ model the second and third quadrant portions of the cross section respectively.

interNET CONNECTION

Graphing Calculator Programs
For a graphing calculator program that determines whether a function is even, odd, or neither, visit **www.amc.glencoe.com**

Functions whose graphs are symmetric with respect to the y-axis are **even functions.** Functions whose graphs are symmetric with respect to the origin are **odd functions.** Some functions are neither even nor odd. From Example 1, $f(x) = x^5$ is an odd function, and $g(x) = \frac{x}{1 - x}$ is neither even nor odd.

even functions	odd functions
$f(-x) = f(x)$	$f(-x) = -f(x)$
 	 
symmetric with respect to the y-axis	symmetric with respect to the origin

GRAPHING CALCULATOR EXPLORATION

You can use the **TRACE** function to investigate the symmetry of a function.

- Graph the function.
- Use TRACE to observe the relationship between points of the graph having opposite x-coordinates.
- Use this information to determine the relationship between $f(x)$ and $f(-x)$.

TRY THESE **Graph each function to determine how $f(x)$ and $f(-x)$ are related.**

1. $f(x) = x^8 - 3x^4 + 2x^2 + 2$
2. $f(x) = x^7 + 4x^5 - x^3$

WHAT DO YOU THINK?

3. Identify the functions in Exercises 1 and 2 as *odd, even,* or *neither* based on your observations of their graphs.

4. Verify your conjectures algebraically.

5. How could you use symmetry to help you graph an even or odd function? Give an example.

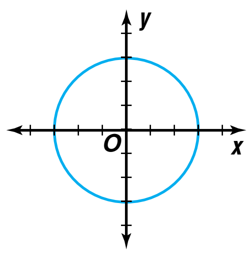
6. Explain how you could use the **ASK** option in **TBLSET** to determine the relationship between $f(x)$ and $f(-x)$ for a given function.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. Refer to the tables on pages 129–130. **Identify** each graph as an even function, an odd function, or neither. Explain.
2. **Explain** how rotating a graph of an odd function 180° will affect its appearance. Draw an example.
3. Consider the graph at the right.
   a. **Determine** four lines of symmetry for the graph.
   b. How many other lines of symmetry does this graph possess?
   c. What other type of symmetry does this graph possess?
4. **Write** an explanation of how to test for symmetry with respect to the line $y = -x$.



5. **You Decide** Alicia says that any graph that is symmetric to the origin and to the y-axis must also be symmetric to the x-axis. Chet disagrees. Who is correct? Support your answer graphically and algebraically.

Guided Practice

Determine whether the graph of each function is symmetric with respect to the origin.

6. $f(x) = x^6 + 9x$

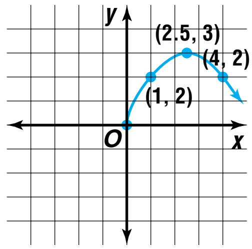
7. $f(x) = \frac{1}{5x} - x^{19}$

Determine whether the graph of each equation is symmetric with respect to the x-axis, y-axis, the line $y = x$, the line $y = -x$, or none of these.

8. $6x^2 = y - 1$

9. $x^3 + y^3 = 4$

10. Copy and complete the graph at the right so that it is the graph of an even function.

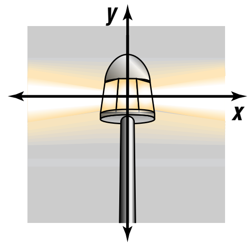


Determine whether the graph of each equation is symmetric with respect to the x-axis, the y-axis, both, or neither. Use the information about symmetry to graph the relation.

11. $y = \sqrt{2 - x^2}$

12. $|y| = x^3$

13. **Physics** Suppose the light pattern from a fog light can be modeled by the equation $\frac{x^2}{25} - \frac{y^2}{9} = 1$. One of the points on the graph of this equation is at $\left(6, \frac{3\sqrt{11}}{5}\right)$, and one of the x-intercepts is -5. Find the coordinates of three additional points on the graph and the other x-intercept.



EXERCISES

Practice

Determine whether the graph of each function is symmetric with respect to the origin.

14. $f(x) = 3x$

15. $f(x) = x^3 - 1$

16. $f(x) = 5x^2 + 6x + 9$

17. $f(x) = \frac{1}{4x^7}$

18. $f(x) = -7x^5 + 8x$

19. $f(x) = \frac{1}{x} - x^{100}$

20. Is the graph of $g(x) = \frac{x^2 - 1}{x}$ symmetric with respect to the origin? Explain how you determined your answer.

Determine whether the graph of each equation is symmetric with respect to the x-axis, y-axis, the line $y = x$, the line $y = -x$, or none of these.

21. $xy = -5$

22. $x + y^2 = 1$

23. $y = -8x$

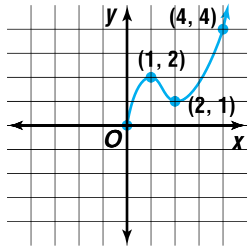
24. $y = \frac{1}{x^2}$

25. $x^2 + y^2 = 4$

26. $y^2 = \frac{4x^2}{9} - 4$

27. Which line(s) are lines of symmetry for the graph of $x^2 = \frac{1}{y^2}$?

For Exercises 28–30, refer to the graph.



28. Complete the graph so that it is the graph of an odd function.

29. Complete the graph so that it is the graph of an even function.

30. Complete the graph so that it is the graph of a function that is neither even nor odd.

Determine whether the graph of each equation is symmetric with respect to the *x*-axis, the *y*-axis, both, or neither. Use the information about symmetry to graph the relation.

31. $y^2 = x^2$

32. $|x| = -3y$

33. $y^2 + 3x = 0$

34. $|y| = 2x^2$

35. $x = \pm\sqrt{12 - 8y^2}$

36. $|y| = xy$

37. Graph the equation $|y| = x^3 - x$ using information about the symmetry of the graph.

Applications and Problem Solving

Real World Application

38. **Physics** The path of a comet around the Sun can be modeled by a transformation of the equation $\frac{x^2}{8} + \frac{y^2}{10} = 1$.

a. Determine the symmetry in the graph of the comet's path.

b. Use symmetry to graph the equation $\frac{x^2}{8} + \frac{y^2}{10} = 1$.

c. If it is known that the comet passes through the point at $(2, \sqrt{5})$, name the coordinates of three other points through which it must pass.

39. **Critical Thinking** Write the equation of a graph that is symmetric with respect to the *x*-axis.

40. **Geometry** Draw a diagram composed of line segments that exhibits both *x*- and *y*-axis symmetry. Write equations for the boundaries.

41. **Communication** Radio waves emitted from two different radio towers interfere with each other's signal. The path of interference can be modeled by the equation $\frac{y^2}{12} - \frac{x^2}{16} = 1$, where the origin is the midpoint of the line segment between the two towers and the positive *y*-axis represents north. Juana lives on an east-west road 6 miles north of the *x*-axis and cannot receive the radio station at her house. At what coordinates might Juana live relative to the midpoint between the two towers?

42. **Critical Thinking** Must the graph of an odd function contain the origin? Explain your reasoning and illustrate your point with the graph of a specific function.

Mixed Review

43. **Manufacturing** A manufacturer makes a profit of \$6 on a bicycle and \$4 on a tricycle. Department A requires 3 hours to manufacture the parts for a bicycle and 4 hours to manufacture parts for a tricycle. Department B takes 5 hours to assemble a bicycle and 2 hours to assemble a tricycle. How many bicycles and tricycles should be produced to maximize the profit if the total time available in department A is 450 hours and in department B is 400 hours? *(Lesson 2-7)*

Extra Practice See p. A30.

44. Find AB if $A = \begin{bmatrix} 4 & 3 \\ 7 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 5 \\ 9 & 6 \end{bmatrix}$. *(Lesson 2-3)*

45. Solve the system of equations, $2x + y + z = 0$, $3x - 2y - 3z = -21$, and $4x + 5y + 3z = -2$. *(Lesson 2-2)*

46. State whether the system, $4x - 2y = 7$ and $-12x + 6y = -21$, is *consistent and independent, consistent and dependent,* or *inconsistent.* *(Lesson 2-1)*

47. Graph $0 \leq x - y \leq 2$. *(Lesson 1-8)*

48. Write an equation in slope-intercept form for the line that passes through $A(0, 2)$ and $B(-2, 16)$. *(Lesson 1-4)*

49. If $f(x) = -2x + 11$ and $g(x) = x - 6$, find $[f \circ g](x)$ and $[g \circ f](x)$. *(Lesson 1-2)*

50. **SAT/ACT Practice** What is the product of 75^3 and 75^7?

A 75^5 **B** 75^{10} **C** 150^{10} **D** 5625^{10} **E** 75^{21}

CAREER CHOICES

Biomedical Engineering

Would you like to help people live better lives? Are you interested in a career in the field of health? If you answered yes, then biomedical engineering may be the career for you. Biomedical engineers apply engineering skills and life science knowledge to design artificial parts for the human body and devices for investigating and repairing the human body. Some examples are artificial organs, pacemakers, and surgical lasers.

In biomedical engineering, there are three primary work areas: research, design, and teaching. There are also many specialty areas in this field. Some of these are bioinstrumentation, biomechanics, biomaterials, and rehabilitation engineering.

The graph shows an increase in the number of outpatient visits over the number of hospital visits. This is due in part to recent advancements in biomedical engineering.

CAREER OVERVIEW

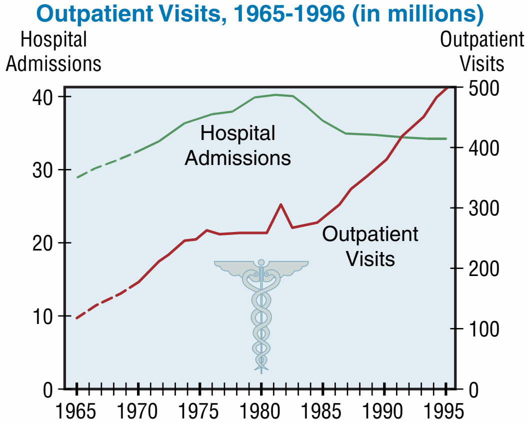
Degree Preferred:
bachelor's degree in biomedical engineering

Related Courses:
biology, chemistry, mathematics

Outlook:
number of jobs expected to increase through the year 2006

Hospital Vital Signs

Total Number of Hospital Admissions and Outpatient Visits, 1965-1996 (in millions)



Source: *The Wall Street Journal Almanac*

interNET CONNECTION For more information on careers in biomedical engineering, visit: **www.amc.glencoe.com**

3-2 Families of Graphs

OBJECTIVES
- Identify transformations of simple graphs.
- Sketch graphs of related functions.

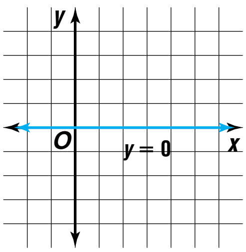
Real World Application

ENTERTAINMENT At some circuses, a human cannonball is shot out of a special cannon. In order to perform this death-defying feat safely, the maximum height and distance of the performer must be calculated accurately. Quadratic functions can be used to model the height of a projectile like a human cannonball at any time during its flight. The quadratic equation used to model height versus time is closely related to the equation of $y = x^2$. *A problem related to this is solved in Example 5.*

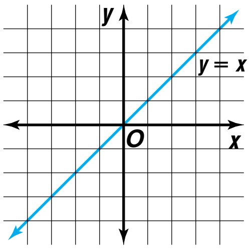
All parabolas are related to the graph of $y = x^2$. This makes $y = x^2$ the **parent graph** of the family of parabolas. Recall that a *family of graphs* is a group of graphs that displays one or more similar characteristics.

A parent graph is a basic graph that is transformed to create other members in a family of graphs. Some different types are shown below. Notice that with the exception of the constant function, the coefficient of x in each equation is 1.

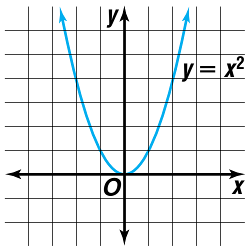
constant function

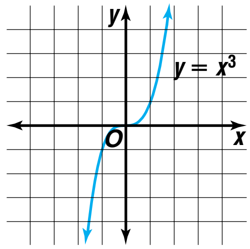


identity function

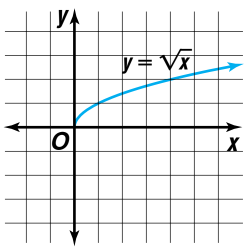


polynomial functions

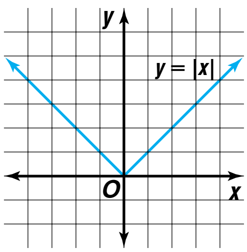




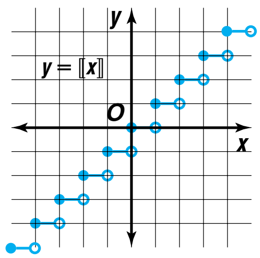
square root function



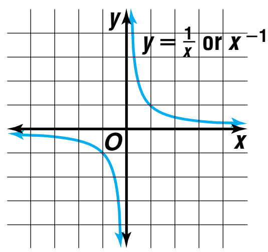
absolute value function



greatest integer function



rational function



Look Back
Refer to Lesson 2-4 for more about reflections and translations.

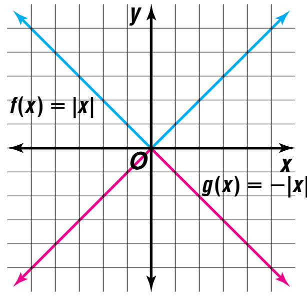
Reflections and translations of the parent function can affect the appearance of the graph. The transformed graph may appear in a different location, but it will resemble the parent graph. A reflection flips a figure over a line called the *axis of symmetry*. *The axis of symmetry is also called the line of symmetry.*

Example 1 **Graph $f(x) = |x|$ and $g(x) = -|x|$. Describe how the graphs of $f(x)$ and $g(x)$ are related.**

Graphing Calculator Tip

You can use a graphing calculator to check your sketch of any function in this lesson.

x	$f(x) = \|x\|$	$g(x) = -\|x\|$
-2	2	-2
-1	1	-1
0	0	0
1	1	-1
2	2	-2



To graph both equations on the same axis, let $y = f(x)$ and $y = g(x)$.

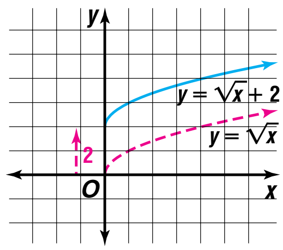
The graph of $g(x)$ is a reflection of the graph of $f(x)$ over the x-axis. The symmetric relationship can be stated algebraically by $g(x) = -f(x)$, or $f(x) = -g(x)$. *Notice that the effect of multiplying a function by -1 is a reflection over the x-axis.*

When a constant c is added to or subtracted from a parent function, the result, $f(x) \pm c$, is a translation of the graph up or down. When a constant c is added or subtracted from x before evaluating a parent function, the result, $f(x \pm c)$, is a translation left or right.

Example 2 **Use the parent graph $y = \sqrt{x}$ to sketch the graph of each function.**

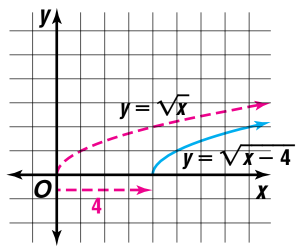
a. $y = \sqrt{x} + 2$

This function is of the form $y = f(x) + 2$. Since 2 is added to the parent function $y = \sqrt{x}$, the graph of the parent function moves up 2 units.



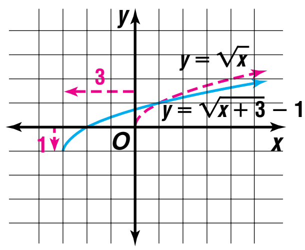
b. $y = \sqrt{x - 4}$

This function is of the form $y = f(x - 4)$. Since 4 is being subtracted from x before being evaluated by the parent function, the graph of the parent function $y = \sqrt{x}$ slides 4 units right.



c. $y = \sqrt{x + 3} - 1$

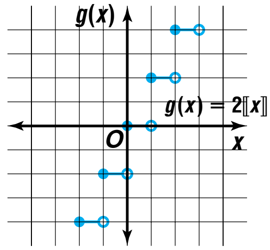
This function is of the form $y = f(x + 3) - 1$. The addition of 3 indicates a slide of 3 units left, and the subtraction of 1 moves the parent function $y = \sqrt{x}$ down 1 unit.



Remember that a dilation has the effect of shrinking or enlarging a figure. Likewise, when the leading coefficient of x is not 1, the function is expanded or compressed.

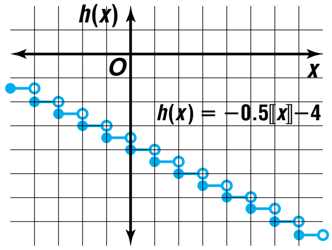
Example 3 **Graph each function. Then describe how it is related to its parent graph.**

a. $g(x) = 2[\![x]\!]$



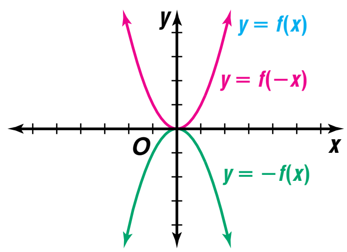
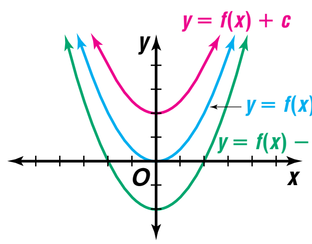
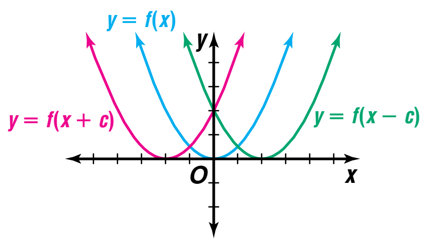
The parent graph is the greatest integer function, $f(x) = [\![x]\!]$. $g(x) = 2[\![x]\!]$ is a vertical expansion by a factor of 2. The vertical distance between the steps is 2 units.

b. $h(x) = -0.5[\![x]\!] - 4$

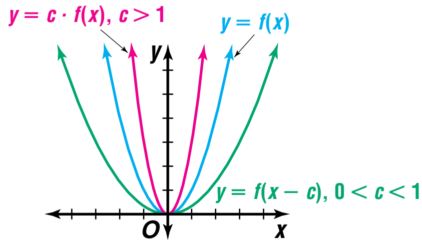
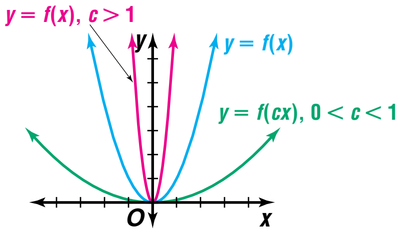


$h(x) = -0.5[\![x]\!] - 4$ reflects the parent graph over the x-axis, compresses it vertically by a factor of 0.5, and shifts the graph down 4 units. *Notice that multiplying by a positive number less than 1 compresses the graph vertically.*

The following chart summarizes the relationships in families of graphs. The parent graph may differ, but the transformations of the graphs have the same effect. Remember that more than one transformation may affect a parent graph.

Change to the Parent Function $y = f(x)$, $c > 0$	Change to Parent Graph	Examples
Reflections $y = -f(x)$ $y = f(-x)$	Is reflected over the x-axis. Is reflected over the y-axis.	
Translations $y = f(x) + c$ $y = f(x) - c$	Translates the graph c units up. Translates the graph c units down.	
$y = f(x + c)$ $y = f(x - c)$	Translates the graph c units left. Translates the graph c units right.	

(continued on the next page)

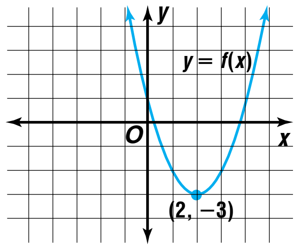
Change to the Parent Function $y = f(x)$, $c > 0$	Change to Parent Graph	Examples
Dilations		
$y = c \cdot f(x)$, $c > 1$ $y = c \cdot f(x)$, $0 < c < 1$	Expands the graph vertically. Compresses the graph vertically.	
$y = f(cx)$, $c > 1$ $y = f(cx)$, $0 < c < 1$	Compresses the graph horizontally. Expands the graph horizontally.	

Other transformations may affect the appearance of the graph. We will look at two more transformations that change the shape of the graph.

Example 4 **Observe the graph of each function. Describe how the graphs in parts b and c relate to the graph in part a.**

a. $f(x) = (x - 2)^2 - 3$

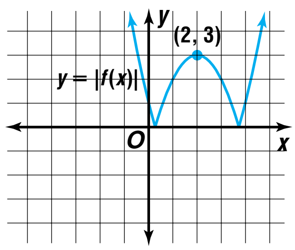
The graph of $y = f(x)$ is a translation of $y = x^2$. The parent graph has been translated right 2 units and down 3 units.



b. $y = |f(x)|$

$|f(x)| = |(x - 2)^2 - 3|$

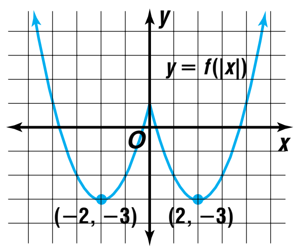
This transformation reflects any portion of the parent graph that is below the x-axis so that it is above the x-axis.



c. $y = f(|x|)$

$f(|x|) = (|x| - 2)^2 - 3$

This transformation results in the portion of the parent graph on the left of the y-axis being replaced by a reflection of the portion on the right of the y-axis.



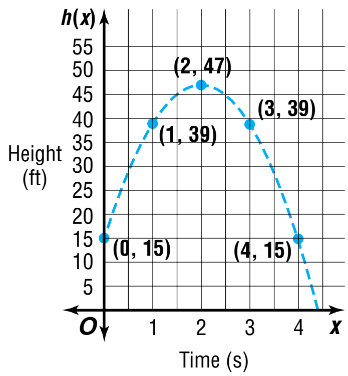
Example 5 Real World Application

ENTERTAINMENT **A traveling circus invites local schools to send math and science teams to its Science Challenge Day. One challenge is to write an equation that most accurately predicts the height of the flight of a human cannonball performer at any given time. Students collect data by witnessing a performance and examining time-lapse photographs of the flight. Using the performer's initial height of 15 feet and the photographs, one team records the data at the right. Write the equation of the related parabola that models the data.**

Time (seconds)	Height (feet)
0	15
1	39
2	47
3	39
4	15

A graph of the data reveals that a parabola is the best model for the data. The parent graph of a parabola is the graph of the equation $y = x^2$. To write the equation of the related parabola that models the data, we need to compare points located near the vertex of each graph. An analysis of the transformation these points have undergone will help us determine the equation of the transformed parabola.

From the graph, we can see that parent graph has been turned upside-down, indicating that the equation for this parabola has been multiplied by some negative constant c. Through further inspection of the graph and its data points, we can see that the vertex of the parent graph has been translated to the point (2, 47). Therefore, an equation that models the data is $y = c(x - 2)^2 + 47$.



To find c, compare points near the vertex of the graphs of the parent function $f(x)$ and the graph of the data points. Look at the relationship between the differences in the y-coordinates for each set of points.

x	$f(x) = x^2$	
−2	4	
−1	1	−3
0	0	−1
1	1	−1
2	4	−3

x Time (seconds)	$h(x)$ Height (feet)	
0	15	
1	39	24
2	47	8
3	39	8
4	15	24

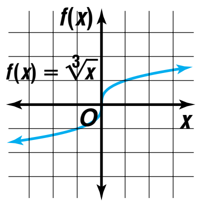
These differences are in a ratio of 1 to −8. This means that the graph of the parent graph has been expanded by a factor of −8. Thus, an equation that models the data is $y = -8(x - 2)^2 + 47$.

CHECK FOR UNDERSTANDING

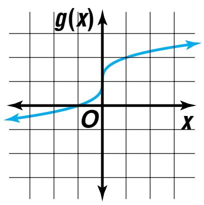
Communicating Mathematics

Read and study the lesson to answer each question.

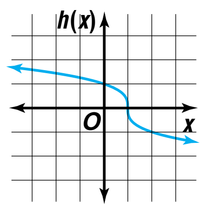
1. **Write** the equation of the graph obtained when the parent graph $y = x^3$ is translated 4 units left and 7 units down.
2. **Explain** the difference between the graphs of $y = (x + 3)^2$ and $y = x^2 + 3$.
3. **Name** two types of transformations for which the pre-image and the image are congruent figures.
4. **Describe** the differences between the graphs of $y = f(x)$ and $y = f(cx)$ for $c > 0$.
5. **Write** equations for the graphs of $g(x)$, $h(x)$, and $k(x)$ if the graph of $f(x) = \sqrt[3]{x}$ is the parent graph.



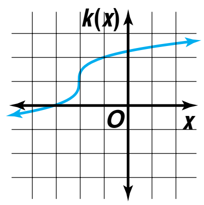
a.



b.



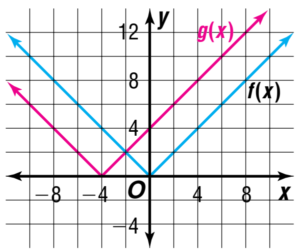
c.



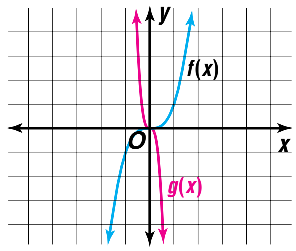
Guided Practice

Describe how the graphs of $f(x)$ and $g(x)$ are related.

6. $f(x) = |x|$ and $g(x) = |x + 4|$



7. $f(x) = x^3$ and $g(x) = -(3x)^3$



Use the graph of the given parent function to describe the graph of each related function.

8. $f(x) = x^2$
   a. $y = (0.2x)^2$
   b. $y = (x - 5)^2 - 2$
   c. $y = 3x^2 + 6$

9. $f(x) = x^3$
   a. $y = |x^3 + 3|$
   b. $y = -(2x)^3$
   c. $y = 0.75(x + 1)^3$

Sketch the graph of each function.

10. $f(x) = 2(x - 3)^3$

11. $g(x) = (0.5x)^2 - 1$

12. **Consumer Costs** The cost of labor for servicing cars at B & B Automotive is \$50 for each whole hour or for any fraction of an hour.
    a. Graph the function that describes the cost for x hours of labor.
    b. Graph the function that would show a \$25 additional charge if you decide to also get the oil changed and fluids checked.
    c. What would be the cost of servicing a car that required 3.45 hours of labor if the owner requested that the oil be changed and the fluids be checked?

EXERCISES

Practice

Describe how the graphs of $f(x)$ and $g(x)$ are related.

13. $f(x) = x$ and $g(x) = x + 6$

14. $f(x) = x^2$ and $g(x) = \frac{3}{4}x^2$

15. $f(x) = |x|$ and $g(x) = |5x|$

16. $f(x) = x^3$ and $g(x) = (x - 5)^3$

17. $f(x) = \frac{1}{x}$ and $g(x) = \frac{3}{x}$

18. $f(x) = [\![x]\!] + 1$ and $g(x) = -[\![x]\!] - 1$

19. Describe the relationship between the graphs of $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{0.4x} + 3$.

Use the graph of the given parent function to describe the graph of each related function.

20. $f(x) = x^2$
- **a.** $y = -(1.5x)^2$
- **b.** $y = 4(x - 3)^2$
- **c.** $y = \frac{1}{2}x^2 - 5$

21. $f(x) = |x|$
- **a.** $y = |0.2x|$
- **b.** $y = 7|x| - 0.4$
- **c.** $y = -9|x + 1|$

22. $f(x) = x^3$
- **a.** $y = (x + 2)^3 - 5$
- **b.** $y = -(0.8x)^3$
- **c.** $y = \left(\frac{5}{3}x\right)^3 + 2$

23. $f(x) = \sqrt{x}$
- **a.** $y = \frac{1}{3}\sqrt{x + 2}$
- **b.** $y = \sqrt{-x} - 7$
- **c.** $y = 4 + 2\sqrt{x - 3}$

24. $f(x) = \frac{1}{x}$
- **a.** $y = \frac{1}{0.5x}$
- **b.** $y = \frac{1}{6x} + 8$
- **c.** $y = \frac{1}{|x|}$

25. $f(x) = [\![x]\!]$
- **a.** $y = \left[\!\!\left[\frac{5}{2}x\right]\!\!\right] - 3$
- **b.** $y = -0.75[\![x]\!]$
- **c.** $y = [\![\,|x| - 4]\!]$

26. Name the parent graph of $m(x) = |-9 + (0.75x)^2|$. Then sketch the graph of $m(x)$.

27. Write the equation of the graph obtained when the graph of $y = \frac{1}{x}$ is compressed vertically by a factor of 0.25, translated 4 units right, and then translated 3 units up.

Sketch the graph of each function.

28. $f(x) = -(x + 4) + 5$

29. $g(x) = |x^2 - 4|$

30. $h(x) = (0.5x - 1)^3$

31. $n(x) = -2.5[\![x]\!] + 3$

32. $q(x) = -4|x - 2| - 1$

33. $k(x) = -\frac{1}{2}(x - 3)^2 - 4$

34. Graph $y = f(x)$ and $y = f(|x|)$ on the same set of axes if $f(x) = (x + 3)^2 - 8$.

Graphing Calculator

Use a graphing calculator to graph each set of functions on the same screen. Name the x-intercept(s) of each function.

35.
- **a.** $y = x^2$
- **b.** $y = (4x - 2)^2$
- **c.** $y = (2x + 3)^2$

36.
- **a.** $y = x^3$
- **b.** $y = (3x - 2)^3$
- **c.** $y = (4x + 1)^3$

37.
- **a.** $y = \sqrt{x}$
- **b.** $y = \sqrt{2x + 5}$
- **c.** $y = \sqrt{5x - 3}$

Applications and Problem Solving

38. **Technology** Transformations can be used to create a series of graphs that appear to move when shown sequentially on a video screen. Suppose you start with the graph of $y = f(x)$. Describe the effect of graphing the following functions in succession if n has integer values from 1 to 100.

a. $y_n = f(x + 2n) - 3n$.

b. $y_n = (-1)^n f(x - n)$.

39. **Critical Thinking** Study the coordinates of the x-intercepts you found in the related graphs in Exercises 35–37. Make a conjecture about the x-intercept of $y = (ax + b)^n$ if $y = x^n$ is the parent function.

40. **Business** The standard cost of a taxi fare is \$1.50 for the first unit and 25 cents for each additional unit. A unit is composed of distance (one unit equals 0.2 mile) and/or wait time (one unit equals 75 seconds). As the cab moves at more than 9.6 miles per hour, the taxi's meter clocks distance. When the cab is stopped or moving at less than 9.6 miles per hour, the meter clocks time. Thus, traveling 0.1 mile and then waiting at a stop light for 37.5 seconds generates one unit and a 25-cent charge.

a. Assuming that the cab meter rounds up to the nearest unit, write a function that would determine the cost for x units of cab fare, where $x > 0$.

b. Graph the function found in part a.

41. **Geometry** Suppose $f(x) = 5 - |x - 6|$.

a. Sketch the graph of $f(x)$ and calculate the area of the triangle formed by $f(x)$ and the positive x-axis.

b. Sketch the graph of $y = 2f(x)$ and calculate the area of the new triangle formed by $2f(x)$ and the positive x-axis. How do the areas of part a and part b compare? Make a conjecture about the area of the triangle formed by $y = c \cdot f(x)$ in the first quadrant if $c \geq 0$.

c. Sketch the graph of $y = f(x - 3)$ and recalculate the area of the triangle formed by $f(x - 3)$ and the positive x-axis. How do the areas of part a and part c compare? Make a conjecture about the area of the triangle formed by $y = f(x - c)$ in the first quadrant if $c \geq 0$.

42. **Critical Thinking** Study the parent graphs at the beginning of this lesson.

a. Select a parent graph or a modification of a parent graph that meets each of the following specifications.
(1) positive at its leftmost points and positive at its rightmost points
(2) negative at its leftmost points and positive at its rightmost points
(3) negative at its leftmost points and negative at its rightmost points
(4) positive at its leftmost points and negative at its rightmost points

b. Sketch the related graph for each parent graph that is translated 3 units right and 5 units down.

c. Write an equation for each related graph.

43. **Critical Thinking** Suppose a reflection, a translation, or a dilation were applied to an even function.

a. Which transformations would result in another even function?

b. Which transformations would result in a function that is no longer even?

Mixed Review

44. Is the graph of $f(x) = x^{17} - x^{15}$ symmetric with respect to the origin? Explain. *(Lesson 3-1)*

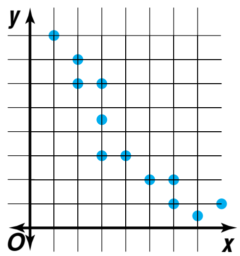
45. **Child Care** Elisa Dimas is the manager for the Learning Loft Day Care Center. The center offers all day service for preschool children for \$18 per day and after school only service for \$6 per day. Fire codes permit only 50 children in the building at one time. State law dictates that a child care worker can be responsible for a maximum of 3 preschool children and 5 school-age children at one time. Ms. Dimas has ten child care workers available to work at the center during the week. How many children of each age group should Ms. Dimas accept to maximize the daily income of the center? *(Lesson 2-7)*

46. **Geometry** Triangle ABC is represented by the matrix $\begin{bmatrix} 5 & 1 & -2 \\ -4 & 3 & -1 \end{bmatrix}$. Find the image of the triangle after a 90° counterclockwise rotation about the origin. *(Lesson 2-4)*

47. Find the values of x, y, and z for which $\begin{bmatrix} x^2 & 7 & 9 \\ 5 & 12 & 6 \end{bmatrix} = \begin{bmatrix} 25 & 7 & y \\ 5 & 2z & 6 \end{bmatrix}$ is true. *(Lesson 2-2)*

48. Solve the system of equations algebraically. *(Lesson 2-1)*
$6x + 5y = -14$
$5x + 2y = -3$

49. Describe the linear relationship implied in the scatter plot at the right. *(Lesson 1-6)*



50. Find the slope of a line perpendicular to a line whose equation is $3x - 4y = 0$. *(Lesson 1-5)*

51. **Fund-Raising** The Band Boosters at Palermo High School are having their annual doughnut sale to raise money for new equipment. The equation $5d - 2p = 500$ represents the amount of profit p in dollars the band will make selling d boxes of doughnuts. What is the p-intercept of the line represented by this equation? *(Lesson 1-3)*

52. Find $[f \circ g](x)$ and $[g \circ f](x)$ if $f(x) = \frac{2}{3}x - 2$ and $g(x) = x^2 - 6x + 9$. *(Lesson 1-2)*

53. **SAT/ACT Practice** If $d = m - \frac{50}{m}$ and m is a positive number that increases in value, then d

A increases in value.
B increases, then decreases.
C remains unchanged.
D decreases in value.
E decreases, then increases.

Extra Practice See p. A30.

3-3

Graphs of Nonlinear Inequalities

OBJECTIVES

- Graph polynomial, absolute value, and radical inequalities in two variables.
- Solve absolute value inequalities.

Real World Application

PHARMACOLOGY Pharmacists label medication as to how much and how often it should be taken. Because oral medication requires time to take effect, the amount of medication in your body varies with time. Suppose the equation $m(x) = 0.5x^4 + 3.45x^3 - 96.65x^2 + 347.7x$ for $0 < x \leq 6$ models the number of milligrams of a certain pain reliever in the bloodstream x hours after taking 400 milligrams of it. The medicine is to be taken every 4 hours. At what times during the first 4-hour period is the level of pain reliever in the bloodstream above 300 milligrams? *This problem will be solved in Example 5.*

Problems like the one above can be solved by graphing inequalities. Graphing inequalities in two variables identifies all ordered pairs that will satisfy the inequality.

Example 1 **Determine whether (3, −4), (4, 7), (1, 1), and (−1, 6) are solutions for the inequality $y \leq (x - 2)^2 - 3$.**

Substitute the x-value and y-value from each ordered pair into the inequality.

$$\begin{aligned} y &\leq (x-2)^2 - 3 \\ -4 &\overset{?}{\leq} (3-2)^2 - 3 \quad (x, y) = (3, -4) \\ -4 &\leq -2 \checkmark \quad \text{true} \end{aligned}$$

$$\begin{aligned} y &\leq (x-2)^2 - 3 \\ 1 &\overset{?}{\leq} (1-2)^2 - 3 \quad (x, y) = (1, 1) \\ 1 &\leq -2 \quad \text{false} \end{aligned}$$

$$\begin{aligned} y &\leq (x-2)^2 - 3 \\ 7 &\overset{?}{\leq} (4-2)^2 - 3 \quad (x, y) = (4, 7) \\ 7 &\leq 1 \quad \text{false} \end{aligned}$$

$$\begin{aligned} y &\leq (x-2)^2 - 3 \\ 6 &\overset{?}{\leq} (-1-2)^2 - 3 \quad (x, y) = (-1, 6) \\ 6 &\leq 6 \checkmark \quad \text{true} \end{aligned}$$

Of these ordered pairs, (3, −4) and (−1, 6) are solutions for $y \leq (x - 2)^2 - 3$.

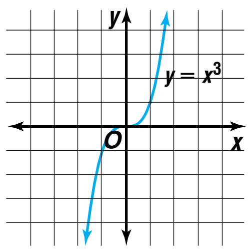
Look Back
Refer to Lesson 1-8 for more about graphing linear inequalities.

Similar to graphing linear inequalities, the first step in graphing nonlinear inequalities is graphing the boundary. You can use concepts from Lesson 3-2 to graph the boundary.

Example 2 **Graph $y \geq (x - 4)^3 - 2$.**

The boundary of the inequality is the graph of $y = (x - 4)^3 - 2$. To graph the boundary curve, start with the parent graph $y = x^3$. Analyze the boundary equation to determine how the boundary relates to the parent graph.

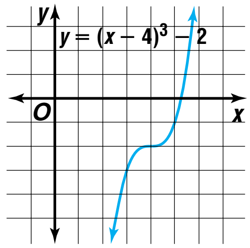
$$y = (x - \underset{\substack{\uparrow \\ \text{move 4 units right}}}{4})^3 - \underset{\substack{\uparrow \\ \text{move 2 units down}}}{2}$$



Graphing Calculator Appendix

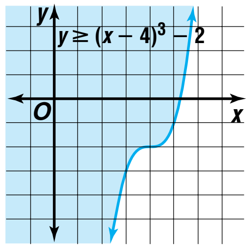
For keystroke instruction on how to graph inequalities see pages A13-A15.

Since the boundary is included in the inequality, the graph is drawn as a solid curve.



The inequality states that the y-values of the solution are greater than the y-values on the graph of $y = (x - 4)^3 - 2$. For a particular value of x, all of the points in the plane that lie above the curve have y-values greater than $y = (x - 4)^3 - 2$. So this portion of the graph should be shaded.

To verify numerically, you can test a point not on the boundary. *It is common to test (0, 0) whenever it is not on the boundary.*



$y \geq (x - 4)^3 - 2$

$0 \stackrel{?}{\geq} (0 - 4)^3 - 2$ *Replace (x, y) with (0, 0).*

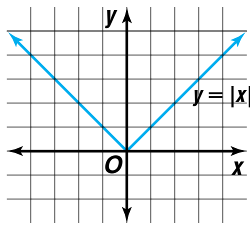
$0 \geq -66$ ✓ *True*

Since (0, 0) satisfies the inequality, the correct region is shaded.

The same process used in Example 2 can be used to graph inequalities involving absolute value.

Example 3 **Graph $y > 3 - |x + 2|$.**

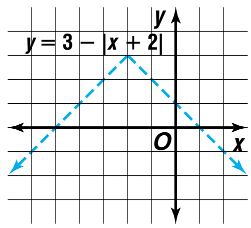
Begin with the parent graph $y = |x|$.



It is easier to sketch the graph of the given inequality if you rewrite it so that the absolute value expression comes first.

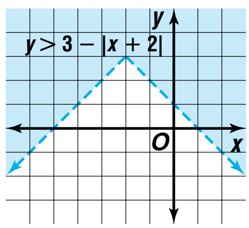
$$y = 3 - |x + 2| \quad \rightarrow \quad y = -|x + 2| + 3$$

This more familiar form tells us the parent graph is reflected over the x-axis and moved 2 units left and three units up. The boundary is not included, so draw it as a dashed line.



The y-values of the solution are greater than the y-values on the graph of $y = 3 - |x + 2|$, so shade above the graph of $y = 3 - |x + 2|$.

Verify by substituting (0, 0) in the inequality to obtain $0 > 1$. Since this statement is false, the part of the graph containing (0, 0) should not be shaded. Thus, the graph is correct.



To solve absolute value inequalities algebraically, use the definition of absolute value to determine the solution set. That is, if $a < 0$, then $|a| = -a$, and if $a \geq 0$, then $|a| = a$.

Example 4 **Solve $|x - 2| - 5 < 4$.**

There are two cases that must be solved. In one case, $x - 2$ is negative, and in the other, $x - 2$ is positive.

Case 1

$$\begin{aligned} (x-2) &< 0 \\ |x-2| - 5 &< 4 \\ -(x-2) - 5 &< 4 \quad |x-2| = -(x-2) \\ -x + 2 - 5 &< 4 \\ -x &< 7 \\ x &> -7 \end{aligned}$$

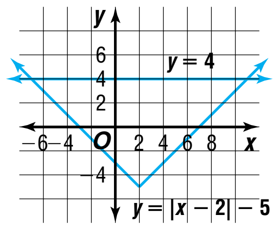
Case 2

$$\begin{aligned} (x-2) &> 0 \\ |x-2| - 5 &< 4 \\ x - 2 - 5 &< 4 \quad |x-2| = (x-2) \\ x - 7 &< 4 \\ x &< 11 \end{aligned}$$

The solution set is $\{x \mid -7 < x < 11\}$. *$\{x \mid -7 < x < 11\}$ is read "the set of all numbers x such that x is between -7 and 11."*

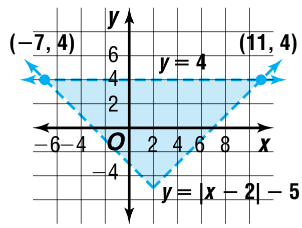
Verify this solution by graphing.

First, graph $y = |x - 2| - 5$. Since we are solving $|x - 2| - 5 < 4$ and $|x - 2| - 5 = y$, we are looking for a region in which $y = 4$. Therefore, graph $y = 4$ and graph it on the same set of axes.



Identify the points of intersection of the two graphs. By inspecting the graph, we can see that they intersect at $(-7, 4)$ and $(11, 4)$.

Now shade the region where the graphs of the inequalities $y > |x - 2| - 5$ and $y < 4$ intersect. This occurs in the region of the graph where $-7 < x < 11$. Thus, the solution to $|x - 2| - 5 < 4$ is the set of x-values such that $-7 < x < 11$.



Nonlinear inequalities have applications to many real-world situations, including business, education, and medicine.

Example 5 Real World Application

PHARMACOLOGY **Refer to the application at the beginning of the lesson. At what times during the first 4-hour period is the amount of pain reliever in the bloodstream above 300 milligrams?**

Since we need to know when the level of pain reliever in the bloodstream is above 300 milligrams, we can write an inequality.

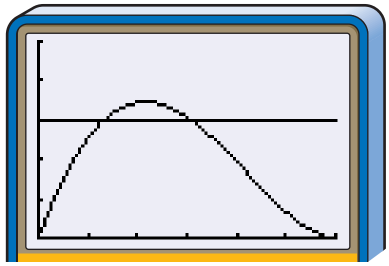
$0.5x^4 + 3.45x^3 - 96.65x^2 + 347.7x > 300$

for $0 < x \leq 6$

Let $y = 0.5x^4 + 3.45x^3 - 96.65x^2 + 347.7x$ and $y = 300$. Graph both equations on the same set of axes using a graphing calculator.

Calculating the points of intersection, we find that the two equations intersect at about (1.3, 300) and (3.1, 300). Therefore, when $1.3 < x < 3.1$, the amount of pain reliever in the bloodstream is above 300 milligrams. That is, the amount exceeds 300 milligrams between about 1 hour 18 minutes and 3 hours 6 minutes after taking the medication.

Weight (mg)



Time (h)

[0, 6] scl: 1 by [0, 500] scl: 100

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** how knowledge of transformations can help you graph the inequality $y \leq 5 + \sqrt{x - 2}$.
2. **State** the two cases considered when solving a one-variable absolute value inequality algebraically.
3. **Write** a procedure for determining which region of the graph of an inequality should be shaded.
4. *Math Journal* **Sketch** the graphs of $y = |x - 3| + 2$ and $y = 1$ on the same set of axes. Use your sketch to solve the inequality $|x - 3| + 2 < 1$. If no solution exists, write *no solution.* Write a paragraph to explain your answer.

Guided Practice

Determine whether the ordered pair is a solution for the given inequality. Write *yes* or *no.*

5. $y \geq -5x^4 + 7x^3 + 8$, $(-1, -3)$
6. $y < |3x - 4| - 1$, $(0, 3)$

Graph each inequality.

7. $y \leq (x + 1)^3$
8. $y \leq 2(x - 3)^2$
9. $y > -|x - 4| + 2$

Solve each inequality.

10. $|x + 6| > 4$
11. $|3x - 4| \leq x$

12. **Manufacturing** The AccuData Company makes compact disks measuring 12 centimeters in diameter. The diameters of the disks can vary no more than 50 micrometers or 5×10^{-3} centimeter.

a. Write an absolute value inequality to represent the range of diameters of compact disks.
b. What are the largest and smallest diameters that are allowable?

EXERCISES

Practice

Determine whether the ordered pair is a solution for the given inequality. Write *yes* or *no*.

13. $y < x^3 - 4x^2 + 2$, $(1, 0)$

14. $y < |x - 2| + 7$, $(3, 8)$

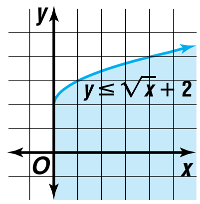
15. $y > -\sqrt{x + 11} + 1$, $(-2, -1)$

16. $y < -0.2x^2 + 9x - 7$, $(10, 63)$

17. $y \leq \dfrac{x^2 - 6}{x}$, $(-6, -9)$

18. $y \geq 2|x|^3 - 7$, $(0, 0)$

19. Which of the ordered pairs, $(0, 0)$, $(1, 4)$, $(1, 1)$, $(-1, 0)$, and $(1, -1)$, is a solution for $y \leq \sqrt{x} + 2$? How can you use these results to determine if the graph at the right is correct?



Graph each inequality.

20. $y \leq x^2 - 4$

21. $y > \sqrt{0.5x}$

22. $y < |x - 9|$

23. $y > |2x| + 3$

24. $y < (x - 5)^2$

25. $y \geq -x^3$

26. $y > -(0.4x)^2$

27. $y \leq |3(x - 4)|$

28. $y < \sqrt{x + 3} + 5$

29. $y \geq (x - 1)^2 - 3$

30. $y \geq (2x + 1)^3 + 2$

31. $y \leq -3|x - 2| + 4$

32. Sketch the graph of the inequality $y \geq x^3 - 6x^2 + 12x - 8$.

Solve each inequality.

33. $|x + 4| > 5$

34. $|3x + 12| \geq 42$

35. $|7 - 2x| - 8 < 3$

36. $|5 - x| \leq x$

37. $|5x - 8| < 0$

38. $|2x + 9| - 2x \geq 0$

39. Find all values of x that satisfy $-\frac{2}{3}|x + 5| \geq -8$.

Applications and Problem Solving

Real World Application

40. Chemistry Katie and Wes worked together on a chemistry lab. They determined the quantity of the unknown in their sample to be 37.5 ± 1.2 grams. If the actual quantity of unknown is x, write their results as an absolute value inequality. Solve for x to find the range of possible values of x.

41. Critical Thinking Solve $3|x - 7| < |x - 1|$.

42. Critical Thinking Find the area of the region described by $y \geq 2|x - 3| + 4$ and $x - 2y \geq -20$.

43. Education Amanda's teacher calculates grades using a weighted average. She counts homework as 10%, quizzes as 15%, projects as 20%, tests as 40%, and the final exam as 15% of the final grade. Going into the final, Amanda has scores of 90 for homework, 75 for quizzes, 76 for projects, and 80 for tests. What grade does Amanda need on the final exam if she wants to get an overall grade of at least an 80?

44. Critical Thinking Consider the equation $|(x - 3)^2 - 4| = b$. Determine the value(s) of b so that the equation has

a. no solution.

b. one solution.

c. two solutions.

d. three solutions.

e. four solutions.

45. **Business** After opening a cookie store in the mall, Paul and Carol Mason hired an consultant to provide them with information on increasing their profit. The consultant told them that their profit P depended on the number of cookies x that they sold according to the relation $P(x) \leq -0.005(x - 1200)^2 + 400$. They typically sell between 950 and 1000 cookies in a given day.

a. Sketch a graph to model this situation.

b. Explain the significance of the shaded region.

Mixed Review

46. How are the graphs of $f(x) = x^3$ and $g(x) = -2x^3$ related? *(Lesson 3-2)*

47. Determine whether the graph of $y = -\frac{1}{x^4}$ is symmetric with respect to the x-axis, y-axis, the line $y = x$, the line $y = -x$, or none of these. *(Lesson 3-1)*

48. Find the inverse of $\begin{bmatrix} 8 & -3 \\ 4 & -5 \end{bmatrix}$. *(Lesson 2-5)*

49. Multiply $\begin{bmatrix} 8 & -7 \\ -4 & 0 \end{bmatrix}$ by $\frac{3}{4}$. *(Lesson 2-3)*

50. Graph $y = 3|x| + 5$. *(Lesson 1-7).*

51. **Criminal Justice** The table shows the number of states with teen courts over a period of several years. Make a scatter plot of the data. *(Lesson 1-6)*

Year	States with Teen Courts
1976	2
1991	14
1994	17
1997	36
1999	47*

Source: American Probation and Parole Association.

*Includes District of Columbia

52. Find $[f \circ g](4)$ and $[g \circ f](4)$ for $f(x) = 5x + 9$ and $g(x) = 0.5x - 1$. *(Lesson 1-2)*

53. **SAT Practice** **Grid-In** Student A is 15 years old. Student B is one-third older. How many years ago was student B twice as old as student A?

MID-CHAPTER QUIZ

Determine whether each graph is symmetric with respect to the x-axis, the y-axis, the line $y = x$, the line $y = -x$, the origin, or none of these. (Lesson 3-1)

1. $x^2 + y^2 - 9 = 0$
2. $5x^2 + 6x - 9 = y$
3. $x = \frac{7}{y}$
4. $y = |x| + 1$

Use the graph of the given parent function to describe the graph of each related function. (Lesson 3-2)

5. $f(x) = [\![x]\!]$
   a. $y = [\![x]\!] - 2$
   b. $y = -[\![x - 3]\!]$
   c. $y = \frac{1}{4}[\![x]\!] + 1$

6. $f(x) = x^3$
   a. $y = 3x^3$
   b. $y = (0.5x)^3 - 1$
   c. $y = (x + 1)^3 + 4$

7. Sketch the graph of $g(x) = -0.5(x - 2)^2 + 3$. (Lesson 3-2)

8. Graph the inequality $y \geq \left(\frac{1}{3}x\right)^2 + 2$. (Lesson 3-3)

9. Find all values of x that satisfy $|2x - 7| < 15$. (Lesson 3-3)

10. **Technology** In September of 1999, a polling organization reported that 64% of Americans were "not very" or "not at all" concerned about the Year-2000 computer bug, with a margin of error of 3%. Write and solve an absolute value inequality to describe the range of the possible percent of Americans who were relatively unconcerned about the "Y2K bug." (Lesson 3-3)

Source: The Gallup Organization

3-4 Inverse Functions and Relations

OBJECTIVES
- Determine inverses of relations and functions.
- Graph functions and their inverses.

Real World Application

METEOROLOGY The hottest temperature ever recorded in Montana was 117° F on July 5, 1937. To convert this temperature to degrees Celsius C, subtract 32° from the Fahrenheit temperature F and then multiply the result by $\frac{5}{9}$. The formula for this conversion is $C = \frac{5}{9}(F - 32)$. The coldest temperature ever recorded in Montana was −57° C on January 20, 1954. To convert this temperature to Fahrenheit, multiply the Celsius temperature by $\frac{9}{5}$ and then add 32°. The formula for this conversion is $F = \frac{9}{5}C + 32$.

The temperature conversion formulas are examples of **inverse functions.** Relations also have inverses, and these inverses are themselves relations.

Inverse Relations	Two relations are inverse relations if and only if one relation contains the element (b, a) whenever the other relation contains the element (a, b).

If $f(x)$ denotes a function, then $f^{-1}(x)$ denotes the inverse of $f(x)$. However, $f^{-1}(x)$ may not necessarily be a function. To graph a function or relation and its inverse, you switch the x- and y-coordinates of the ordered pairs of the function. This results in graphs that are symmetric to each other with respect to the line $y = x$.

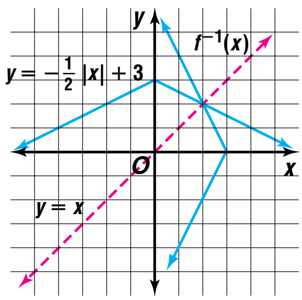
Example 1 **Graph $f(x) = -\frac{1}{2}|x| + 3$ and its inverse.**

To graph the function, let $y = f(x)$. To graph $f^{-1}(x)$, interchange the x- and y-coordinates of the ordered pairs of the function.

Note that the domain of one relation or function is the range of the inverse and vice versa.

$f(x) = -\frac{1}{2}\lvert x\rvert + 3$	
x	$f(x)$
−3	1.5
−2	2
−1	2.5
0	3
1	2.5
2	2
3	1.5

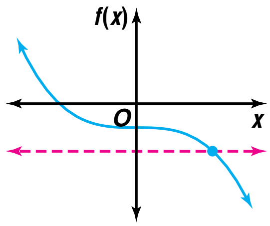
$f^{-1}(x)$	
x	$f^{-1}(x)$
1.5	−3
2	−2
2.5	−1
3	0
2.5	1
2	2
1.5	3



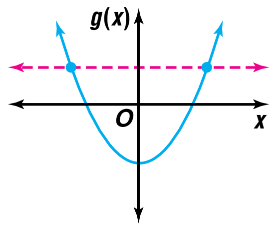
The graph of $f^{-1}(x)$ is the reflection of $f(x)$ over the line $y = x$.

Note that the inverse in Example 1 is not a function because it fails the vertical line test.

You can use the **horizontal line test** to determine if the inverse of a relation will be a function. If every horizontal line intersects the graph of the relation in at most one point, then the inverse of the relation is a function.



The inverse of f(x) is a function.



The inverse of g(x) is not a function.

You can find the inverse of a relation algebraically. First, let $y = f(x)$. Then interchange x and y. Finally, solve the resulting equation for y.

Example 2 **Consider $f(x) = (x + 3)^2 - 5$.**

a. Is the inverse of $f(x)$ a function?

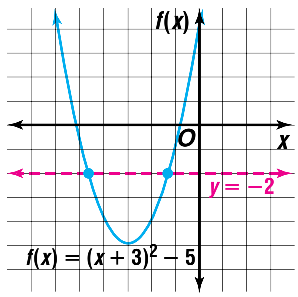
b. Find $f^{-1}(x)$.

c. Graph $f(x)$ and $f^{-1}(x)$ using a graphing calculator.

Graphing Calculator Tip

You can differentiate the appearance of your graphs by highlighting the symbol in front of each equation in the **Y=** list and pressing ENTER to select line (\\), thick (****), or dot (⋱).

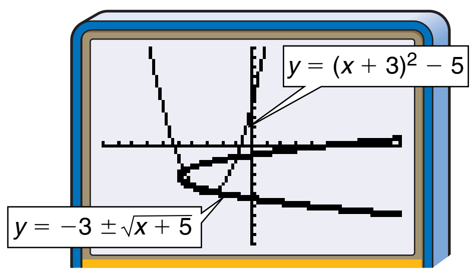
a. Since the line $y = -2$ intersects the graph of $f(x)$ at more than one point, the function fails the horizontal line test. Thus, the inverse of $f(x)$ is not a function.



b. To find $f^{-1}(x)$, let $y = f(x)$ and interchange x and y. Then, solve for y.

$y = (x + 3)^2 - 5$	*Let $y = f(x)$.*
$x = (y + 3)^2 - 5$	*Interchange x and y.*
$x + 5 = (y + 3)^2$	*Isolate the expression containing y.*
$\pm\sqrt{x + 5} = y + 3$	*Take the square root of each side.*
$y = -3 \pm \sqrt{x + 5}$	*Solve for y.*
$f^{-1}(x) = -3 \pm \sqrt{x + 5}$	*Replace y with $f^{-1}(x)$.*

c. To graph $f(x)$ and its inverse, enter the equations $y = (x + 3)^2 - 5$, $y = -3 + \sqrt{x + 5}$, and $y = -3 - \sqrt{x + 5}$ in the same viewing window.

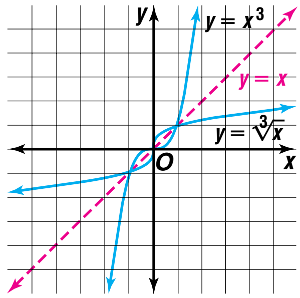


[−15.16, 15.16] scl: 2 by [−10, 10] scl: 2

You can graph a function by using the parent graph of an inverse function.

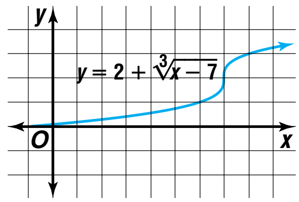
Example 3 **Graph $y = 2 + \sqrt[3]{x - 7}$.**

The parent function is $y = \sqrt[3]{x}$ which is the inverse of $y = x^3$.



To graph $y = \sqrt[3]{x}$, start with the graph of $y = x^3$. Reflect the graph over the line $y = x$.

To graph $y = 2 + \sqrt[3]{x - 7}$, translate the reflected graph 7 units to the right and 2 units up.



To find the inverse of a function, you use an **inverse process** to solve for y after switching variables. This inverse process can be used to solve many real-world problems.

Example 4 Real World Application

FINANCE **When the Garcias decided to begin investing, their financial advisor instructed them to set a goal. Their net pay is about 65% of their gross pay. They decided to subtract their monthly food allowance from their monthly net pay and then invest 10% of the remainder.**

a. Write an equation that gives the amount invested I as a function of their monthly gross pay G given that they allow \$450 per month for food.

b. Determine the equation for the inverse process and describe the real-world situation it models.

c. Determine the gross pay needed in order to invest \$100 per month.

a. One model for the amount they will invest is as follows.

investment	*equals*	*10%*	*of*	(*65% of gross pay*	*less*	*food allowance*)
↓	↓	↓	↓	↓	↓	↓
I	$=$	0.10	$\cdot$	$(0.65G$	$-$	$450)$

b. Solve for G.

$$I = 0.10(0.65G - 450)$$

$$10I = 0.65G - 450 \quad \textit{Multiply each side by 10.}$$

$$10I + 450 = 0.65G \quad \textit{Add 450 to each side.}$$

$$\frac{10I + 450}{0.65} = G \quad \textit{Divide each side by 0.65.}$$

This equation models the gross pay needed to meet a monthly investment goal I with the given conditions.

c. Substituting 100 for I gives $G = \frac{10I + 450}{0.65}$, or about \$2231. So the Garcias need to earn a monthly gross pay of about \$2231 in order to invest \$100 per month.

Look Back
Refer to Lesson 1-2 to review the composition of two functions.

If the inverse of a function is also a function, then a composition of the function and its inverse produces a unique result.

Consider $f(x) = 3x - 2$ and $f^{-1}(x) = \frac{x+2}{3}$. You can find functions $[f \circ f^{-1}](x)$ and $[f^{-1} \circ f](x)$ as follows.

$$
\begin{aligned}
[f \circ f^{-1}](x) &= f\left(\frac{x+2}{3}\right) && f^{-1}(x) = \frac{x+2}{3} \\
&= 3\left(\frac{x+2}{3}\right) - 2 && f(x) = 3x - 2 \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
[f^{-1} \circ f](x) &= f^{-1}(3x - 2) && f(x) = 3x - 2 \\
&= \frac{(3x-2)+2}{3} && f^{-1}(x) = \frac{x+2}{3} \\
&= x
\end{aligned}
$$

This leads to the formal definition of inverse functions.

Inverse Functions	Two functions, f and f^{-1}, are inverse functions if and only if $[f \circ f^{-1}](x) = [f^{-1} \circ f](x) = x$.

Example 5 **Given $f(x) = 4x - 9$, find $f^{-1}(x)$, and verify that f and f^{-1} are inverse functions.**

$$
\begin{aligned}
y &= 4x - 9 && f(x) = y \\
x &= 4y - 9 && \text{Interchange } x \text{ and } y. \\
x + 9 &= 4y && \text{Solve for } y. \\
\frac{x+9}{4} &= y \\
\frac{x+9}{4} &= f^{-1}(x) && \text{Replace } y \text{ with } f^{-1}(x).
\end{aligned}
$$

Now show that $[f \circ f^{-1}](x) = [f^{-1} \circ f](x) = x$.

$$
\begin{aligned}
[f \circ f^{-1}](x) &= f\left(\frac{x+9}{4}\right) \\
&= 4\left(\frac{x+9}{4}\right) - 9 \\
&= x
\end{aligned}
\qquad
\begin{aligned}
[f^{-1} \circ f](x) &= f^{-1}(4x - 9) \\
&= \frac{(4x-9)+9}{4} \\
&= x
\end{aligned}
$$

Since $[f \circ f^{-1}](x) = [f^{-1} \circ f](x) = x$, f and f^{-1} are inverse functions.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** an explanation of how to determine the equation for the inverse of the relation $y = \pm\sqrt{x - 3}$.
2. **Determine** the values of n for which $f(x) = x^n$ has an inverse that is a function. Assume that n is a whole number.

3. **Find a counterexample** to this statement: The inverse of a function is also a function.

4. **Show** how you know whether the inverse of a function is also a function without graphing the inverse.

5. **You Decide** Nitayah says that the inverse of $y = 3 \pm \sqrt{x+2}$ cannot be a function because $y = 3 \pm \sqrt{x+2}$ is not a function. Is she right? Explain.

Guided Practice

Graph each function and its inverse.

6. $f(x) = |x| + 1$
7. $f(x) = x^3 + 1$
8. $f(x) = -(x-3)^2 + 1$

Find $f^{-1}(x)$. Then state whether $f^{-1}(x)$ is a function.

9. $f(x) = -3x + 2$
10. $f(x) = \frac{1}{x^3}$
11. $f(x) = (x+2)^2 + 6$

12. Graph the equation $y = 3 \pm \sqrt{x+1}$ using the parent graph $p(x) = x^2$.

13. Given $f(x) = \frac{1}{2}x - 5$, find $f^{-1}(x)$. Then verify that f and f^{-1} are inverse functions.

14. **Finance** If you deposit \$1000 in a savings account with an interest rate of r compounded annually, then the balance in the account after 3 years is given by the function $B(r) = 1000(1+r)^3$, where r is written as a decimal.
   a. Find a formula for the interest rate, r, required to achieve a balance of B in the account after 3 years.
   b. What interest rate will yield a balance of \$1100 after 3 years?

EXERCISES

Practice

Graph each function and its inverse.

15. $f(x) = |x| + 2$
16. $f(x) = |2x|$
17. $f(x) = x^3 - 2$
18. $f(x) = x^5 - 10$
19. $f(x) = [\![x]\!]$
20. $f(x) = 3$
21. $f(x) = x^2 + 2x + 4$
22. $f(x) = -(x+2)^2 - 5$
23. $f(x) = (x+1)^2 - 4$
24. For $f(x) = x^2 + 4$, find $f^{-1}(x)$. Then graph $f(x)$ and $f^{-1}(x)$.

Find $f^{-1}(x)$. Then state whether $f^{-1}(x)$ is a function.

25. $f(x) = 2x + 7$
26. $f(x) = -x - 2$
27. $f(x) = \frac{1}{x}$
28. $f(x) = -\frac{1}{x^2}$
29. $f(x) = (x-3)^2 + 7$
30. $f(x) = x^2 - 4x + 3$
31. $f(x) = \frac{1}{x+2}$
32. $f(x) = \frac{1}{(x-1)^2}$
33. $f(x) = -\frac{2}{(x-2)^3}$

34. If $g(x) = \frac{3}{x^2 + 2x}$, find $g^{-1}(x)$.

Graph each equation using the graph of the given parent function.

35. $f(x) = \sqrt{x+5}$, $p(x) = x^2$
36. $y = 1 \pm \sqrt{x-2}$, $p(x) = x^2$
37. $f(x) = -2 - \sqrt[3]{x+3}$, $p(x) = x^3$
38. $y = 2\sqrt[5]{x-4}$, $p(x) = x^5$

Given $f(x)$, find $f^{-1}(x)$. Then verify that f and f^{-1} are inverse functions.

39. $f(x) = -\frac{2}{3}x + \frac{1}{6}$
40. $f(x) = (x-3)^3 + 4$

www.amc.glencoe.com/self_check_quiz

CONTENTS

Applications and Problem Solving

41. **Analytic Geometry** The function $d(x) = |x - 4|$ gives the distance between x and 4 on the number line.
   a. Graph $d^{-1}(x)$.
   b. Is $d^{-1}(x)$ a function? Why or why not?
   c. Describe what $d^{-1}(x)$ represents. Then explain how you could have predicted whether $d^{-1}(x)$ is a function without looking at a graph.

42. **Fire Fighting** The velocity v and maximum height h of water being pumped into the air are related by the equation $v = \sqrt{2gh}$ where g is the acceleration due to gravity (32 feet/second^2).
   a. Determine an equation that will give the maximum height of the water as a function of its velocity.
   b. The Mayfield Fire Department must purchase a pump that is powerful enough to propel water 80 feet into the air. Will a pump that is advertised to project water with a velocity of 75 feet/second meet the fire department's needs? Explain.

43. **Critical Thinking**
   a. Give an example of a function that is its own inverse.
   b. What type of symmetry must the graph of the function exhibit?
   c. Would all functions with this type of symmetry be their own inverses? Justify your response.

44. **Consumer Costs** A certain long distance phone company charges callers 10 cents for every minute or part of a minute that they talk. Suppose that you talk for x minutes, where x is any real number greater than 0.
   a. Sketch the graph of the function $C(x)$ that gives the cost of an x-minute call.
   b. What are the domain and range of $C(x)$?
   c. Sketch the graph of $C^{-1}(x)$.
   d. What are the domain and range of $C^{-1}(x)$?
   e. What real-world situation is modeled by $C^{-1}(x)$?

45. **Critical Thinking** Consider the parent function $y = x^2$ and its inverse $y = \pm\sqrt{x}$. If the graph of $y = x^2$ is translated 6 units right and 5 units down, what must be done to the graph of $y = \pm\sqrt{x}$ to get a graph of the inverse of the translated function? Write an equation for each of the translated graphs.

46. **Physics** The formula for the kinetic energy of a particle as a function of its mass m and velocity v is $KE = \frac{1}{2}mv^2$.
   a. Find the equation for the velocity of a particle based on its mass and kinetic energy.
   b. Use your equation from part a to find the velocity in meters per second of a particle of mass 1 kilogram and kinetic energy 15 joules.
   c. Explain why the velocity of the particle is not a function of its mass and kinetic energy.

47. Cryptography One way to encode a message is to assign a numerical value to each letter of the alphabet and encode the message by assigning each number to a new value using a mathematical relation.

a. Does the encoding relation have to be a function? Explain.

b. Why should the graph of the encoding function pass the horizontal line test?

c. Suppose a value was assigned to each letter of the alphabet so that $1 = \text{A}$, $2 = \text{B}$, $3 = \text{C}$, …, $26 = \text{Z}$, and a message was encoded using the relation $c(x) = -2 + \sqrt{x + 3}$. What function would decode the message?

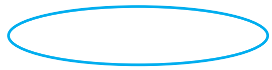
d. Try this decoding function on the following message:

1	2.899	2.123	0.449	2.796	1.464	2.243	2.123	2.690
0	2.583	0.828	1	2.899	2.123			

Mixed Review

48. Solve the inequality $|2x + 4| \leq 6$. *(Lesson 3-3)*

49. State whether the figure at the right has point symmetry, line symmetry, neither, or both. *(Lesson 3-1)*



50. Retail Arturo Alvaré, a sales associate at a paint store, plans to mix as many gallons as possible of colors A and B. He has 32 units of blue dye and 54 units of red dye. Each gallon of color A requires 4 units of blue dye and 1 unit of red dye. Each gallon of color B requires 1 unit of blue dye and 6 units of red dye. Use linear programming to answer the following questions. *(Lesson 2-7)*

a. Let a be the number of gallons of color A and let b be the number of gallons of color B. Write the inequalities that describe this situation.

b. Find the maximum number of gallons possible.

51. Solve the system of equations $4x + 2y = 10$, $y = 6 - x$ by using a matrix equation. *(Lesson 2-5)*

52. Find the product $\frac{1}{2}\begin{bmatrix} 9 & -3 \\ -6 & 6 \end{bmatrix}$. *(Lesson 2-3)*

53. Graph $y < -2x + 8$. *(Lesson 1-8)*

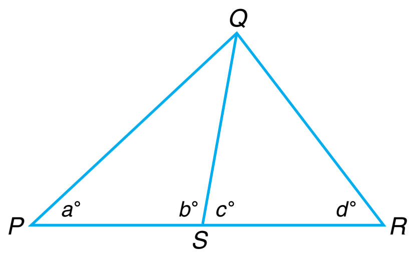
54. Line ℓ_1 has a slope of $\frac{1}{4}$, and line ℓ_2 has a slope of 4. Are the lines *parallel*, *perpendicular*, or *neither*? *(Lesson 1-5)*

55. Write the slope-intercept form of the equation of the line that passes through points at (0, 7) and (5, 2). *(Lesson 1-4)*

56. SAT/ACT Practice In the figure at the right, if $\overline{PQ}$ is perpendicular to $\overline{QR}$, then $a + b + c + d = ?$

A 180 **B** 225 **C** 270

D 300 **E** 360



Extra Practice See p. A30.

CONTENTS

3-5

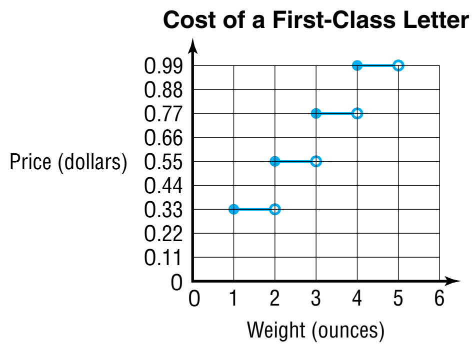
Continuity and End Behavior

OBJECTIVES

- Determine whether a function is continuous or discontinuous.
- Identify the end behavior of functions.
- Determine whether a function is increasing or decreasing on an interval.

Real World Application

POSTAGE On January 10, 1999, the United States Postal Service raised the cost of a first-class stamp. After the change, mailing a letter cost \$0.33 for the first ounce and \$0.22 for each additional ounce or part of an ounce. The graph summarizes the cost of mailing a first-class letter. *A problem related to this is solved in Example 2.*

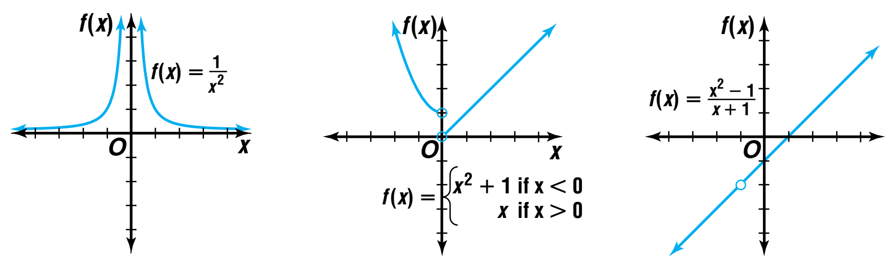


Most graphs we have studied have been smooth, continuous curves. However, a function like the one graphed above is a **discontinuous** function. That is, you cannot trace the graph of the function without lifting your pencil.

There are many types of discontinuity. Each of the functions graphed below illustrates a different type of discontinuity. That is, each function is discontinuous at some point in its domain.

The postage graph exhibits jump discontinuity.

Graphs of Discontinuous Functions

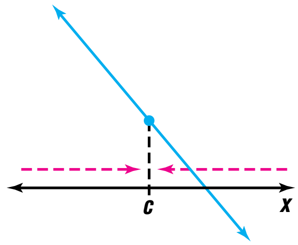


Infinite Discontinuity | Jump Discontinuity | Point Discontinuity

- **Infinite discontinuity** means that $|f(x)|$ becomes greater and greater as the graph approaches a given x-value.
- **Jump discontinuity** indicates that the graph stops at a given value of the domain and then begins again at a different range value for the same value of the domain.
- When there is a value in the domain for which the function is undefined, but the pieces of the graph match up, we say the function has **point discontinuity.**

There are functions that are impossible to graph in the real number system. Some of these functions are said to be **everywhere discontinuous.** An example of such a function is $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ -1 \text{ if } x \text{ is irrational} \end{cases}$.

If a function is not discontinuous, it is said to be **continuous**. That is, a function is continuous at a number c if there is a point on the graph with x-coordinate c and the graph passes through that point without a break.



Linear and quadratic functions are continuous at all points. If we only consider x-values less than c as x approaches c, then we say x is approaching c from the left. Similarly, if we only consider x-values greater than c as x approaches c, then we say x is approaching c from the right.

Continuity Test

A function is continuous at $x = c$ if it satisfies the following conditions:
(1) the function is defined at c; in other words, $f(c)$ exists;
(2) the function approaches the same y-value on the left and right sides of $x = c$; and
(3) the y-value that the function approaches from each side is $f(c)$.

Example 1 **Determine whether each function is continuous at the given x-value.**

a. $f(x) = 3x^2 + 7$; $x = 1$

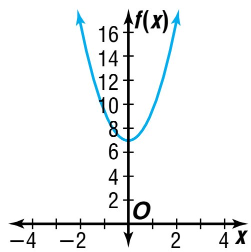
Check the three conditions in the continuity test.

(1) The function is defined at $x = 1$. In particular, $f(1) = 10$.

(2) The first table below suggests that when x is less than 1 and x approaches 1, the y-values approach 10. The second table suggests that when x is greater than 1 and x approaches 1, the y-values approach 10.

x	$y = f(x)$
0.9	9.43
0.99	9.9403
0.999	9.994003

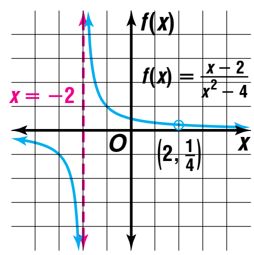
x	$y = f(x)$
1.1	10.63
1.01	10.0603
1.001	10.006003



(3) Since the y-values approach 10 as x approaches 1 from both sides and $f(1) = 10$, the function is continuous at $x = 1$. This can be confirmed by examining the graph.

b. $f(x) = \frac{x-2}{x^2-4}$; $x = 2$

Start with the first condition in the continuity test. The function is not defined at $x = 2$ because substituting 2 for x results in a denominator of zero. So the function is discontinuous at $x = 2$.
This function has point discontinuity at $x = 2$.



c. $f(x) = \begin{cases} \frac{1}{x} \text{ if } x > 1 \\ x \text{ if } x \leq 1 \end{cases}; x = 1$

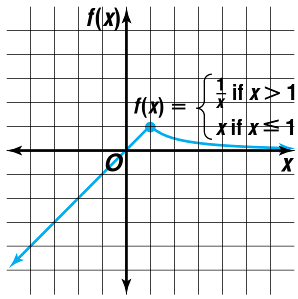
The function is defined at $x = 1$.
Using the second formula we find $f(1) = 1$.

The first table suggests that $f(x)$ approaches 1 as x approaches 1 from the left. We can see from the second table that $f(x)$ seems to approach 1 as x approaches 1 from the right.

x	$f(x)$
0.9	0.9
0.99	0.99
0.999	0.999

x	$f(x)$
1.1	0.9091
1.01	0.9901
1.001	0.9990

Since the $f(x)$-values approach 1 as x approaches 1 from both sides and $f(1) = 1$, the function is continuous at $x = 1$.



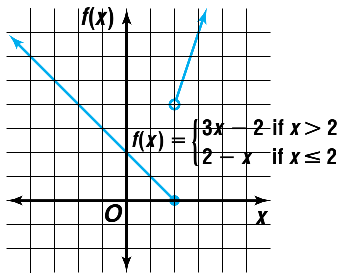
A function may have a discontinuity at one or more x-values but be continuous on an interval of other x-values. For example, the function $f(x) = \frac{1}{x^2}$ is continuous for $x > 0$ and $x < 0$, but discontinuous at $x = 0$.

Continuity on an Interval — A function $f(x)$ is continuous on an interval if and only if it is continuous at each number x in the interval.

In Chapter 1, you learned that a piecewise function is made from several functions over various intervals. The piecewise function

$f(x) = \begin{cases} 3x - 2 \text{ if } x > 2 \\ 2 - x \text{ if } x \leq 2 \end{cases}$ is continuous for $x > 2$

and $x < 2$ but is discontinuous at $x = 2$. The graph has a jump discontinuity. This function fails the second part of the continuity test because the values of $f(x)$ approach 0 as x approaches 2 from the left, but the $f(x)$-values approach 4 as x approaches 2 from the right.



Example 2 **POSTAGE** **Refer to the application at the beginning of the lesson.**

a. Use the continuity test to show that the step function is discontinuous.

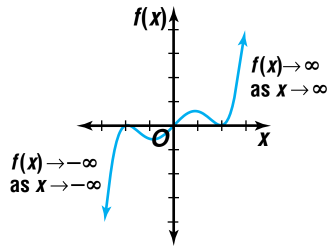
b. Explain why a continuous function would not be appropriate to model postage costs.

a. The graph of the postage function is discontinuous at each integral value of w in its domain because the function does not approach the same value from the left and the right. For example, as w approaches 1 from the left, $C(w)$ approaches 0.33 but as w approaches 1 from the right, $C(w)$ approaches 0.55.

b. A continuous function would have to achieve all real y-values (greater than or equal to 0.33.) This would be an inappropriate model for this situation since the weight of a letter is rounded to the nearest ounce and postage costs are rounded to the nearest cent.

$x \to \infty$ is read as "x approaches infinity."

Another tool for analyzing functions is **end behavior.** The end behavior of a function describes what the y-values do as $|x|$ becomes greater and greater. When x becomes greater and greater, we say that x approaches infinity, and we write $x \to \infty$. Similarly, when x becomes more and more negative, we say that x approaches negative infinity, and we write $x \to -\infty$. The same notation can also be used with y or $f(x)$ and with real numbers instead of infinity.



Example 3 **Describe the end behavior of $f(x) = -2x^3$ and $g(x) = -x^3 + x^2 - x + 5$.**

Use your calculator to create a table of function values so you can investigate the behavior of the y-values.

$f(x) = -2x^3$	
x	**f(x)**
−10,000	2×10^{12}
−1000	2×10^9
−100	2,000,000
−10	2000
0	0
10	−2000
100	−2,000,000
1000	-2×10^9
10,000	-2×10^{12}

$g(x) = x^3 + x^2 - x + 5$	
x	**g(x)**
−10,000	1.0001×10^{12}
−1000	1,001,001,005
−100	1,010,105
−10	1115
0	5
10	−905
100	−990,095
1000	−999,000,995
10,000	-9.999×10^{11}

Notice that both polynomial functions have y-values that become very large in absolute value as x gets very large in absolute value. The end behavior of $f(x)$ can be summarized by stating that as $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to \infty$. The end behavior of $g(x)$ is the same. *You may wish to graph these functions on a graphing calculator to verify this summary.*

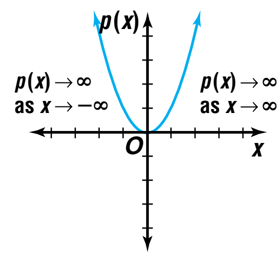
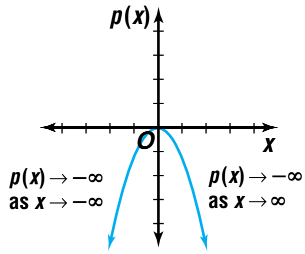
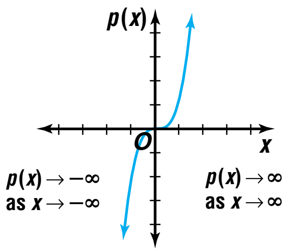
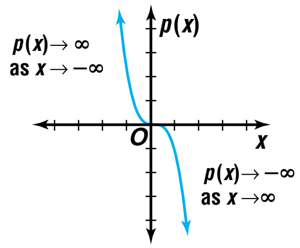
In general, the end behavior of any polynomial function can be modeled by the function comprised solely of the term with the highest power of x and its coefficient. Suppose for $n \geq 0$

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0.$$

Then $f(x) = a_n x^n$ has the same end behavior as $p(x)$. The following table organizes the information for such functions and provides an example of a function displaying each type of end behavior.

End Behavior of Polynomial Functions

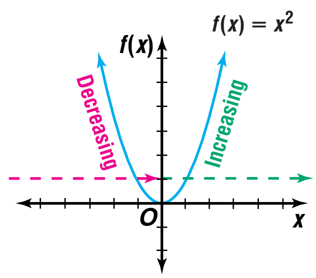
$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0,\ n > 0$$

a_n: positive, n: even	a_n: negative, n: even
$p(x) = x^2$  	$p(x) = -x^2$ 
a_n: positive, n: odd	**a_n: negative, n: odd**
$p(x) = x^3$  	$p(x) = -x^3$ 

Another characteristic of functions that can help in their analysis is the **monotonicity** of the function. A function is said to be monotonic on an interval I if and only if the function is increasing on I or decreasing on I.

Whether a graph is increasing or decreasing is always judged by viewing a graph from left to right.

The graph of $f(x) = x^2$ shows that the function is decreasing for $x < 0$ and increasing for $x > 0$.



Increasing, Decreasing, and Constant Functions	A function f is increasing on an interval I if and only if for every a and b contained in I, $f(a) < f(b)$ whenever $a < b$. A function f is decreasing on an interval I if and only if for every a and b contained in I, $f(a) > f(b)$ whenever $a < b$. A function f remains constant on an interval I if and only if for every a and b contained in I, $f(a) = f(b)$ whenever $a < b$.

Points in the domain of a function where the function changes from increasing to decreasing or vice versa are special points called *critical points.* You will learn more about these special points in Lesson 3-6. Using a graphing calculator can help you determine where the direction of the function changes.

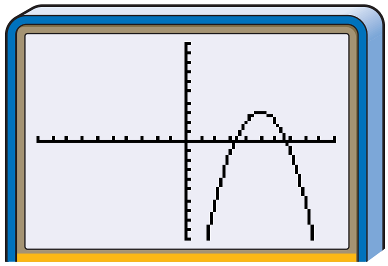
Graphing Calculator Tip

By watching the x- and y-values while using the **TRACE** function, you can determine approximately where a function changes from increasing to decreasing and vice versa.

Example 4 **Graph each function. Determine the interval(s) on which the function is increasing and the interval(s) on which the function is decreasing.**

a. $f(x) = 3 - (x - 5)^2$

The graph of this function is obtained by transforming the parent graph $p(x) = x^2$. The parent graph has been reflected over the x-axis, translated 5 units to the right, and translated up 3 units. The function is increasing for $x < 5$ and decreasing for $x > 5$. At $x = 5$, there is a critical point.



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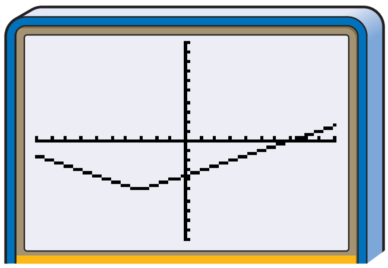
b. $f(x) = \frac{1}{2}|x + 3| - 5$

The graph of this function is obtained by transforming the parent graph $p(x) = |x|$. The parent graph has been vertically compressed by a factor of $\frac{1}{2}$, translated 3 units to the left, and translated down 5 units. This function is decreasing for $x < -3$ and increasing for $x > -3$. There is a critical point when $x = -3$.



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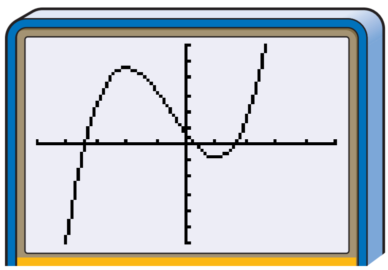
c. $f(x) = 2x^3 + 3x^2 - 12x + 3$

This function has more than one critical point. It changes direction at $x = -2$ and $x = 1$. The function is increasing for $x < -2$. The function is also increasing for $x > 1$. When $-2 < x < 1$, the function is decreasing.



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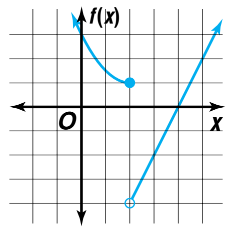
Check for Understanding

Communicating Mathematics

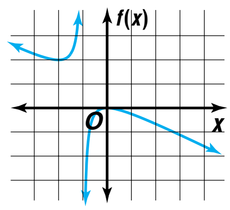
Read and study the lesson to answer each question.

1. **Explain** why the function whose graph is shown at the right is discontinuous at $x = 2$.



2. **Summarize** the end behavior of polynomial functions.

3. **State** whether the graph at the right has infinite discontinuity, jump discontinuity, or point discontinuity, or is continuous. Then describe the end behavior of the function.



4. *Math Journal* **Write** a paragraph that compares the monotonicity of $f(x) = x^2$ with that of $g(x) = -x^2$. In your paragraph, make a conjecture about the monotonicity of the reflection over the x-axis of any function as compared to that of the original function.

Guided Practice

Determine whether each function is continuous at the given x-value. Justify your answer using the continuity test.

5. $y = \dfrac{x-5}{x+3}$; $x = -3$

6. $f(x) = \begin{cases} x^2 + 2 \text{ if } x < -2 \\ 3x \text{ if } x \geq -2 \end{cases}$; $x = -2$

Describe the end behavior of each function.

7. $y = 4x^5 + 2x^4 - 3x - 1$

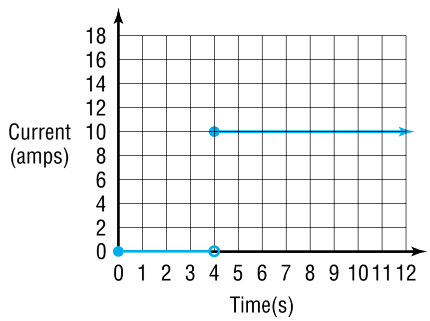
8. $y = -x^6 + x^4 - 5x^2 + 4$

Graph each function. Determine the interval(s) for which the function is increasing and the interval(s) for which the function is decreasing.

9. $f(x) = (x + 3)^2 - 4$

10. $y = \dfrac{x}{x^2 + 1}$

11. **Electricity** A simple electric circuit contains only a power supply and a resistor. When the power supply is off, there is no current in the circuit. When the power supply is turned on, the current almost instantly becomes a constant value. This situation can be modeled by a graph like the one shown at the right. I represents current in amps, and t represents time in seconds.



a. At what t-value is this function discontinuous?

b. When was the power supply turned on?

c. If the person who turned on the power supply left and came back hours later, what would he or she measure the current in the circuit to be?

EXERCISES

Practice

Determine whether each function is continuous at the given *x*-value. Justify your answer using the continuity test.

12. $y = x^3 - 4;\ x = 1$

13. $y = \dfrac{x+1}{x-2};\ x = 2$

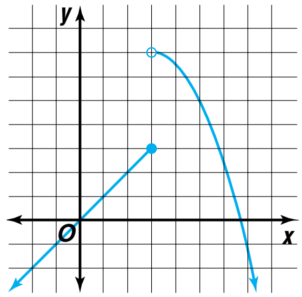
14. $f(x) = \dfrac{x+3}{(x-3)^2};\ x = -3$

15. $y = \left[\!\left[\dfrac{1}{2}x\right]\!\right];\ x = 3$

16. $f(x) = \begin{cases} 3x + 5 \text{ if } x \le -4 \\ -x + 2 \text{ if } x > -4 \end{cases};\ x = -4$

17. $f(x) = \begin{cases} 2x + 1 \text{ if } x \ge 1 \\ 4 - x^2 \text{ if } x < 1 \end{cases};\ x = 1$

18. Determine whether the graph at the right has infinite discontinuity, jump discontinuity, or point discontinuity, or is continuous.



19. Find a value of x at which the function $g(x) = \dfrac{x-4}{x^2 - 3x}$ is discontinuous. Use the continuity test to justify your answer.

Describe the end behavior of each function.

20. $y = x^3 + 2x^2 + x - 1$

21. $y = 8 - x^3 - 2x^4$

22. $f(x) = x^{10} - x^9 + 5x^8$

23. $g(x) = \left|(x-3)^2 - 1\right|$

24. $y = \dfrac{1}{x^2}$

25. $f(x) = -\dfrac{1}{x^3} + 2$

Graphing Calculator

Graph each function. Determine the interval(s) for which the function is increasing and the interval(s) for which the function is decreasing.

26. $y = x^3 + 3x^2 - 9x$

27. $y = -x^3 - 2x + 1$

28. $f(x) = \dfrac{1}{x+1} - 4$

29. $g(x) = \dfrac{x^2 + 5}{x - 2}$

30. $y = \left|x^2 - 4\right|$

31. $y = (2|x| - 3)^2 + 1$

Applications and Problem Solving

32. Physics The gravitational potential energy of an object is given by $U(r) = -\dfrac{GmM_e}{r}$, where G is Newton's gravitational constant, m is the mass of the object, M_e is the mass of Earth, and r is the distance from the object to the center of Earth. What happens to the gravitational potential energy of the object as it is moved farther and farther away from Earth?

33. Critical Thinking A function $f(x)$ is increasing when $0 < x < 2$ and decreasing when $x > 2$. The function has a jump discontinuity when $x = 3$ and $f(x) \to -\infty$ as $x \to \infty$.

a. If $f(x)$ is an even function, then describe the behavior of $f(x)$ for $x < 0$. Sketch a graph of such a function.

b. If $f(x)$ is an odd function, then describe the behavior of $f(x)$ for $x < 0$. Sketch a graph of such a function.

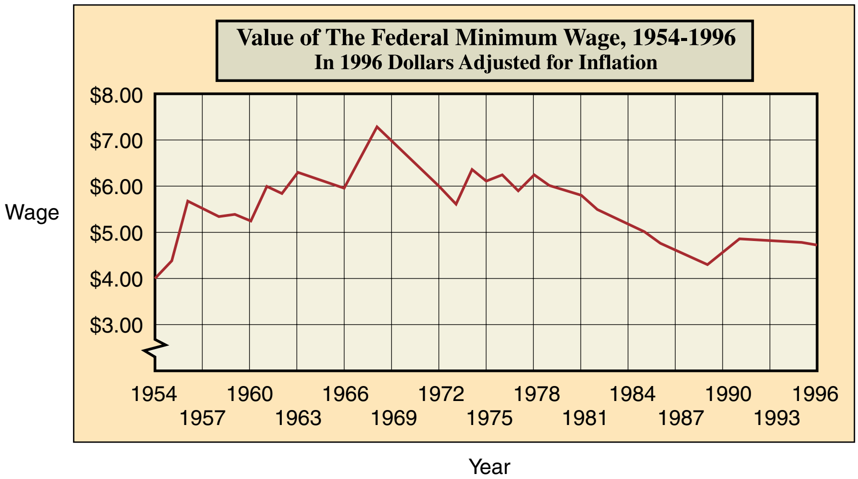
34. **Biology** One model for the population P of bacteria in a sample after t days is given by $P(t) = 1000 - 19.75t + 20t^2 - \frac{1}{3}t^3$.

a. What type of function is $P(t)$?

b. When is the bacteria population increasing?

c. When is it decreasing?

35. **Employment** The graph shows the minimum wage over a 43-year period in 1996 dollars adjusted for inflation.



Source: Department of Labor

interNET CONNECTION

Data Update For the latest information about the minimum wage, visit **www.amc.glencoe.com.**

a. During what time intervals was the adjusted minimum wage increasing?

b. During what time intervals was the adjusted minimum wage decreasing?

36. **Analytic Geometry** A line is *secant* to the graph of a function if it intersects the graph in at least two distinct points. Consider the function $f(x) = -(x - 4)^2 - 3$.

a. On what interval(s) is $f(x)$ increasing?

b. Choose two points in the interval from part a. Determine the slope of the secant line that passes through those two points.

c. Make a conjecture about the slope of any secant line that passes through two points contained in an interval where a function is increasing. Explain your reasoning.

d. On what interval(s) is $f(x)$ decreasing?

e. Extend your hypothesis from part c to describe the slope of any secant line that passes through two points contained in an interval where the function is decreasing. Test your hypothesis by choosing two points in the interval from part d.

37. **Critical Thinking** Suppose a function is defined for all x-values and its graph passes the horizontal line test.

a. What can be said about the monotonicity of the function?

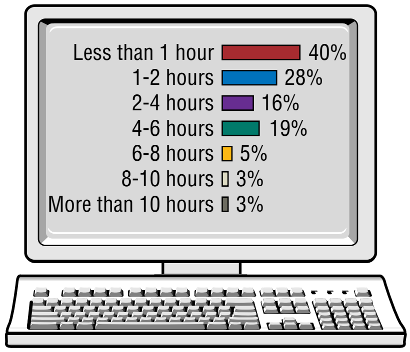
b. What can be said about the monotonicity of the inverse of the function?

38. Computers The graph at the right shows the amount of school computer usage per week for students between the ages of 12 and 18.

a. Use this set of data to make a graph of a step function. On each line segment in your graph, put the open circle at the right endpoint.

b. On what interval(s) is the function continuous?

Student Computer Usage



Source: Consumer Electronics Manufacturers Association

39. Critical Thinking Determine the values of a and b so that f is continuous.

$$f(x) = \begin{cases} x^2 + a \text{ if } x \geq 2 \\ bx + a \text{ if } -2 < x < 2 \\ \sqrt{-b - x} \text{ if } x \leq -2 \end{cases}$$

Mixed Review

40. Find the inverse of the function $f(x) = (x + 5)^2$. *(Lesson 3-4)*

41. Describe how the graphs of $f(x) = |x|$ and $g(x) = |x + 2| - 4$ are related. *(Lesson 3-2)*

42. Find the maximum and minimum values of $f(x, y) = x + 2y$ if it is defined for the polygonal convex set having vertices at (0, 0), (4, 0), (3, 5), and (0, 5). *(Lesson 2-6)*

43. Find the determinant of $\begin{bmatrix} 5 & -4 \\ 8 & 2 \end{bmatrix}$. *(Lesson 2-5)*

44. Consumer Costs Mario's Plumbing Service charges a fee of \$35 for every service call they make. In addition, they charge \$47.50 for every hour they work on each job. *(Lesson 1-4)*

a. Write an equation to represent the cost c of a service call that takes h hours to complete.

b. Find the cost of a $2\frac{1}{4}$-hour service call.

45. Find $f(-2)$ if $f(x) = 2x^2 - 2x + 8$. *(Lesson 1-1)*

46. SAT Practice One box is a cube with side of length x. Another box is a rectangular solid with sides of lengths $x + 1$, $x - 1$, and x. If $x > 1$, how much greater is the volume of the cube than that of the other box?

A x
B $x^2 - 1$
C $x - 1$
D 1
E 0

Extra Practice See p. A31.

CONTENTS

GRAPHING CALCULATOR EXPLORATION

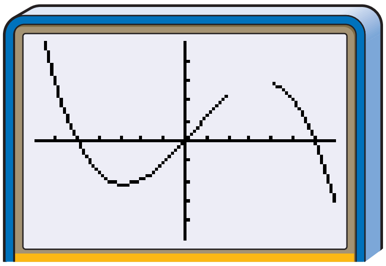
3-5B Gap Discontinuities

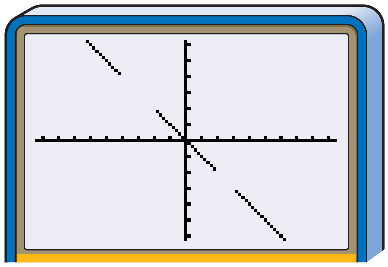
An Extension of Lesson 3-5

OBJECTIVE

- Construct and graph functions with gap discontinuities.

A function has a *gap discontinuity* if there is some interval of real numbers for which it is not defined. The graphs below show two types of gap discontinuities. The first function is undefined for $2 < x < 4$, and the second is undefined for $-4 \leq x \leq -2$ and for $2 \leq x \leq 3$.





These are also called Boolean operators.

The relational and logical operations on the TEST menu are primarily used for programming. Recall that the calculator delivers a value of 1 for a true equation or inequality and a value of 0 for a false equation or inequality. Expressions that use the logical connectives ("and", "or", "not", and so on) are evaluated according to the usual truth-table rules. Enter each of the following expressions and press ENTER to confirm that the calculator displays the value shown.

Expression	Value
$3 \leq 7$	1
$-2 > -5$	1
$-4 > 6$	0

Expression	Value
$(-1 < 8)$ and $(8 < 9)$	1
$(2 > 4)$ or $(-7 \geq -3)$	0
$(4 < 3)$ and $(1 < 12)$	0

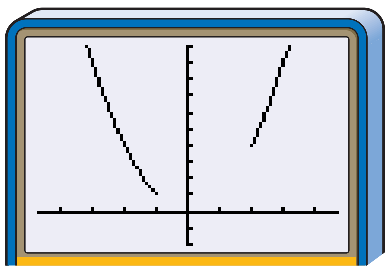
Relational and logical operations are also useful in defining functions that have point and gap discontinuities.

Example 1 **Graph $y = x^2$ for $x \leq -1$ or $x \geq 2$.**

Enter the following expression as **Y1** on the Y= list.

$$X^2/((X \leq -1) \text{ or } (X \geq 2))$$

The function is defined as a quotient. The denominator is a Boolean statement, which has a value of 1 or 0. If the x-value for which the numerator is being evaluated is a member of the interval defined in the denominator, the denominator has a value of 1. Therefore, $\frac{f(x)}{1} = f(x)$ and that part of the function appears on the screen.



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(continued on the next page)

If the x-value is not part of the interval, the value of the denominator is 0. At these points, $\frac{f(x)}{0}$ would be undefined. Thus no graph appears on the screen for this interval.

When you use relational and logical operations to define functions, be careful how you use parentheses. Omitting parentheses can easily lead to an expression that the calculator may interpret in a way you did not intend.

TRY THESE

Graph each function and state its domain. You may need to adjust the window settings.

1. $y = \dfrac{x^2 - 2}{(x > 3)}$

2. $y = \dfrac{0.5x + 1}{((x \geq -2) \text{ and } (x \leq 4))}$

3. $y = \dfrac{-2x}{((x < -3) \text{ or } (x \geq 1))}$

4. $y = \dfrac{-0.2x^3 + 0.3x^2 - x}{((x \leq -3) \text{ or } (x > -2))}$

5. $y = \dfrac{|x|}{(|x| > 1)}$

6. $y = \dfrac{|x - 1| - |x - 3|}{(|x + 4| > 2)}$

7. $y = \dfrac{1.5x}{([\![x]\!] \neq 3)}$

8. $y = \dfrac{0.5x^2}{(([\![x]\!] \neq -2) \text{ and } ([\![x]\!] \neq 1))}$

Relational and logical operations are not the only tools available for defining and graphing functions with gap discontinuities. The square root function can easily be used for such functions. Graph each function and state its domain.

9. $y = \sqrt{(x - 1)(x - 2)(x - 3)(x - 4)}$

10. $y = \dfrac{x}{\dfrac{\sqrt{(x - 1)(x - 2)}}{\sqrt{(x - 1)(x - 2)}}}$

WHAT DO YOU THINK?

11. Suppose you want to construct a function whose graph is like that of $y = x^2$ except for "bites" removed for the values between 2 and 5 and the values between 7 and 8. What equation could you use for the function?

12. Is it possible to use the functions on the **MATH NUM** menu to take an infinite number of "interval bites" from the graph of a function? Justify your answer.

13. Is it possible to write an equation for a function whose graph looks just like the graph of $y = x^2$ for $x \leq -2$ and just like the graph of $y = 2x - 4$ for $x \geq 4$, with no points on the graph for values of x between -2 and 4? Justify your answer.

14. Use what you have learned about gap discontinuities to graph the following piecewise functions.

    a. $f(x) = \begin{cases} -2x \text{ if } x < 0 \\ -x^4 + 2x^3 + 3x^2 + 3x \text{ if } x \geq 0 \end{cases}$

    b. $g(x) = \begin{cases} (x + 4)^3 - 2 \text{ if } x < -2 \\ x^2 + 2 \text{ if } -2 \leq x \leq 2 \\ -(x - 4)^3 - 2 \text{ if } x > 2 \end{cases}$

3-6 Critical Points and Extrema

OBJECTIVE

- Find the extrema of a function.

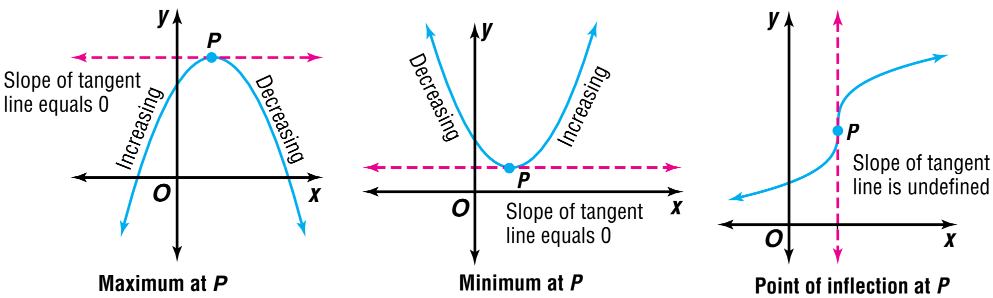
Real World Application

BUSINESS America's 23 million small businesses employ more than 50% of the private workforce. Owning a business requires good management skills. Business owners should always look for ways to compete and improve their businesses. Some business owners hire an analyst to help them identify strengths and weaknesses in their operation. Analysts can collect data and develop mathematical models that help the owner increase productivity, maximize profit, and minimize waste. *A problem related to this will be solved in Example 4.*

Optimization is an application of mathematics where one searches for a maximum or a minimum quantity given a set of constraints. When maximizing or minimizing quantities, it can be helpful to have an equation or a graph of a mathematical model for the quantity to be optimized.

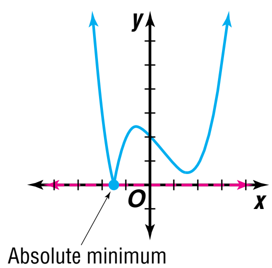
Recall from geometry that a line is tangent to a curve if it intersects a curve in exactly one point.

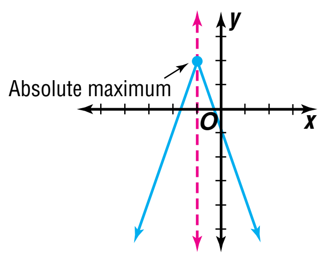
Critical points are those points on a graph at which a line drawn tangent to the curve is horizontal or vertical. A polynomial may possess three types of critical points. A critical point may be a **maximum**, a **minimum**, or a **point of inflection**. When the graph of a function is increasing to the left of $x = c$ and decreasing to the right of $x = c$, then there is a maximum at $x = c$. When the graph of a function is decreasing to the left of $x = c$ and increasing to the right of $x = c$, then there is a minimum at $x = c$. A point of inflection is a point where the graph changes its curvature as illustrated below.



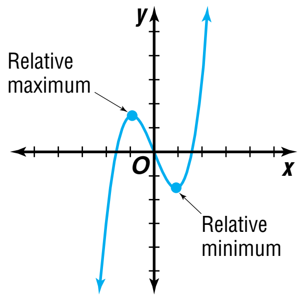
Maxima is the plural of maximum and minima is the plural of minimum. Extrema is the plural of extremum.

The graph of a function can provide a visual clue as to when a function has a maximum or a minimum value. The greatest value that a function assumes over its domain is called the **absolute maximum**. Likewise the least value of a function is the **absolute minimum**. The general term for maximum or minimum is **extremum**. The functions graphed below have absolute extrema.



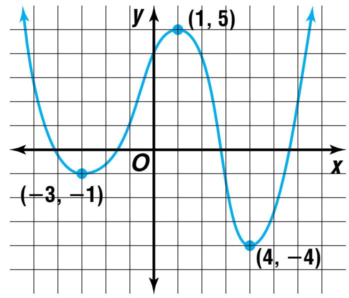


Functions can also have **relative extrema**. A **relative maximum** value of a function may not be the greatest value of f on the domain, but it is the greatest y-value on some interval of the domain. Similarly, a **relative minimum** is the least y-value on some interval of the domain. The function graphed at the right has both a relative maximum and a relative minimum.



Note that extrema are *values of the function;* that is, they are the y-coordinates of each maximum and minimum point.

Example 1 **Locate the extrema for the graph of $y = f(x)$. Name and classify the extrema of the function.**



The function has a relative minimum at $(-3, -1)$.

The function has a relative maximum at $(1, 5)$.

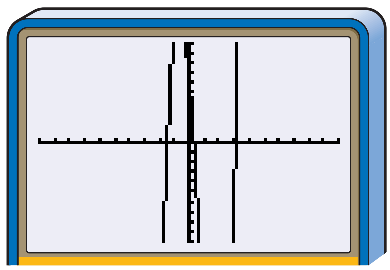
The function has a relative minimum at $(4, -4)$.

Since the point $(4, -4)$ is the lowest point on the graph, the function appears to have an absolute minimum of -4 when $x = 4$. This function appears to have no absolute maximum values, since the graph indicates that the function increases without bound as $x \to \infty$ and as $x \to -\infty$.

A branch of mathematics called calculus can be used to locate the critical points of a function. You will learn more about this in Chapter 15. A graphing calculator can also help you locate the critical points of a polynomial function.

Example 2 **Use a graphing calculator to graph $f(x) = 5x^3 - 10x^2 - 20x + 7$ and to determine and classify its extrema.**

Use a graphing calculator to graph the function in the standard viewing window. Notice that the x-intercepts of the graph are between -2 and -1, 0 and 1, and 3 and 4. Relative maxima and minima will occur somewhere between pairs of x-intercepts.



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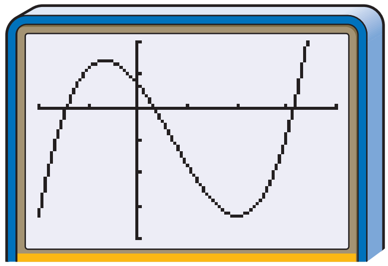
For a better view of the graph of the function, we need to change the window to encompass the observed x-intercepts more closely.

Graphing Calculator Tip

Xmin and **Xmax** define the *x*-axis endpoints. Likewise, **Ymin** and **Ymax** define the *y*-axis endpoints.

One way to do this is to change the *x*-axis view to $-2 \leq x \leq 4$. Since the top and bottom of the graph are not visible, you will probably want to change the *y*-axis view as well. The graph at the right shows $-40 \leq y \leq 20$. From the graph, we can see there is a relative maximum in the interval $-1 < x < 0$ and a relative minimum in the interval $1 < x < 3$.



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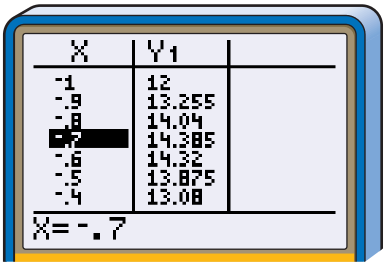
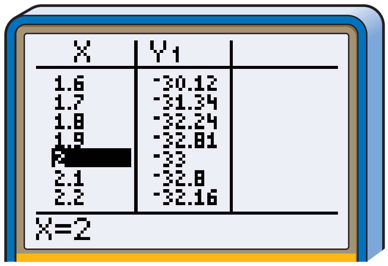
There are several methods you can use to locate these extrema more accurately.

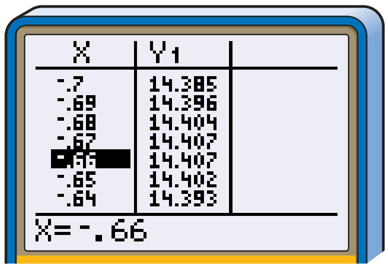
Method 1: Use a table of values to locate the approximate greatest and least value of the function. (*Hint:* Revise **TBLSET** to begin at −2 in intervals of 0.1.)



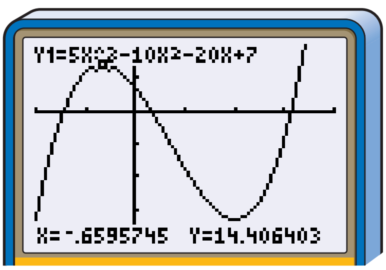


There seems to be a relative maximum of approximately 14.385 at $x = -0.7$ and a relative minimum of −33 at $x = 2$.

You can adjust the **TBLSET** increments to hundredths to more closely estimate the *x*-value for the relative maximum. A relative maximum of about 14.407 appears to occur somewhere between the *x*-values −0.67 and −0.66. A fractional estimation of the *x*-value might be $x = -\frac{2}{3}$.



Method 2: Use the **TRACE** function to approximate the relative maximum and minimum.



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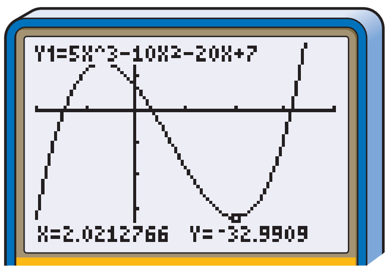

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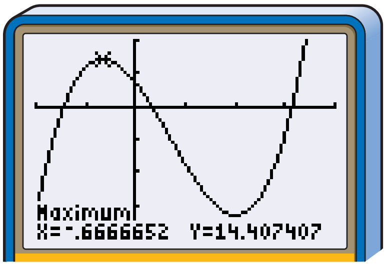
There seems to be a maximum at $x \approx -0.66$ and a minimum at $x \approx 2.02$.

Method 3: Use **3:minimum** and **4:maximum** options on the **CALC** menu to locate the approximate relative maximum and minimum.



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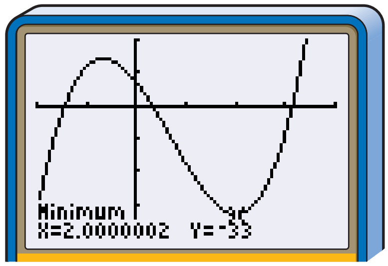

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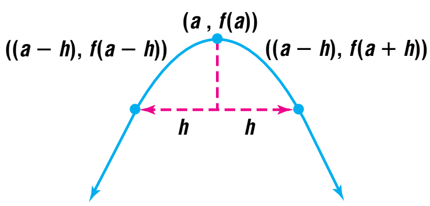
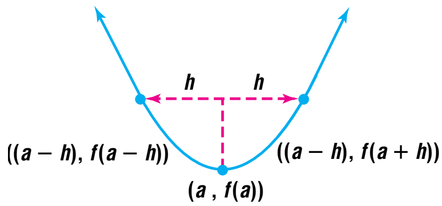
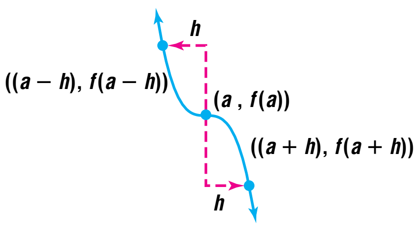
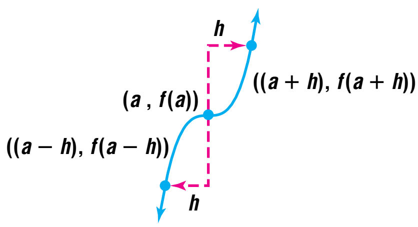
The calculator indicates a relative maximum of about 14.4 at $x \approx -0.67$ and a relative minimum of -33 at $x \approx 2.0$.

All of these approaches give approximations; some more accurate than others. From these three methods, we could estimate that a relative maximum occurs near the point at $(-0.67, 14.4)$ or $\left(-\frac{2}{3}, 14.407\right)$ and a relative minimum near the point at $(2, -33)$.

If you know a critical point of a function, you can determine if it is the location of a relative minimum, a relative maximum, or a point of inflection by testing points on both sides of the critical point. The table below shows how to identify each type of critical point.

Critical Points	
For $f(x)$ with $(a, f(a))$ as a critical point and h as a small value greater than zero	
 $f(a-h) < f(a)$ $f(a+h) < f(a)$ $f(a)$ is a maximum.	 $f(a-h) > f(a)$ $f(a+h) > f(a)$ $f(a)$ is a minimum.
 $f(a-h) > f(a)$ $f(a+h) < f(a)$ $f(a)$ is a point of inflection.	 $f(a-h) < f(a)$ $f(a+h) > f(a)$ $f(a)$ is a point of inflection.

You can also determine whether a critical point is a maximum, minimum, or inflection point by examining the values of a function using a table.

Example 3 **The function $f(x) = 2x^5 - 5x^4 - 10x^3$ has critical points at $x = -1$, $x = 0$, and $x = 3$. Determine whether each of these critical points is the location of a maximum, a minimum, or a point of inflection.**

Evaluate the function at each point. Then check the values of the function around each point. Let $h = 0.1$.

x	$x - 0.1$	$x + 0.1$	$f(x - 0.1)$	$f(x)$	$f(x + 0.1)$	Type of Critical Point
−1	−1.1	−0.9	2.769	3	2.829	maximum
0	−0.1	0.1	0.009	0	−0.010	inflection point
3	2.9	3.1	−187.308	−189	−187.087	minimum

You can verify this solution by graphing f(x) on a graphing calculator.

You can use critical points from the graph of a function to solve real-world problems involving maximization and minimization of values.

Example 4 Real World Application

BUSINESS **A small business owner employing 15 people hires an analyst to help the business maximize profits. The analyst gathers data and develops the mathematical model $P(x) = \frac{1}{3}x^3 - 34x^2 + 1012x$. In this model, P is the owner's monthly profits, in dollars, and x is the number of employees. The model has critical points at $x = 22$ and $x = 46$.**

a. Determine which, if any, of these critical points is a maximum.

b. What does this critical point suggest to the owner about business operations?

c. What are the risks of following the analyst's recommendation?

a. Test values around the points. Let $h = 0.1$.

x	$x - 0.1$	$x + 0.1$	$P(x - 0.1)$	$P(x)$	$P(x + 0.1)$	Type of Critical Point
22	21.9	22.1	9357.21	9357.33	9357.21	maximum
46	45.9	46.1	7053.45	7053.33	7053.45	minimum

The profit will be at a maximum when the owner employs 22 people.

b. The owner should consider expanding the business by increasing the number of employees from 15 to 22.

c. It is important that the owner hire qualified employees. Hiring unqualified employees will likely cause profits to decline.

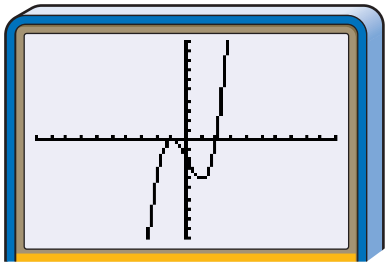
Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** an explanation of how to determine if a critical point is a maximum, minimum, or neither.

2. **Determine** whether the point at $(1, -4)$, a critical point of the graph of $f(x) = x^3 - 3x - 2$ shown at the right, represents a relative maximum, a relative minimum, or a point of inflection. Explain your reasoning.



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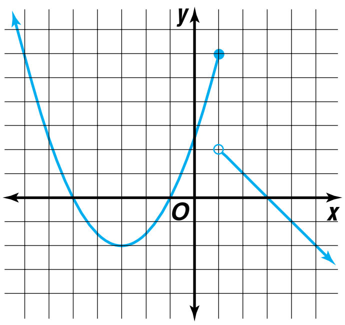
3. **Sketch** the graph of a function that has a relative minimum at $(0, -4)$, a relative maximum at $(-3, 1)$, and an absolute maximum at $(4, 6)$.

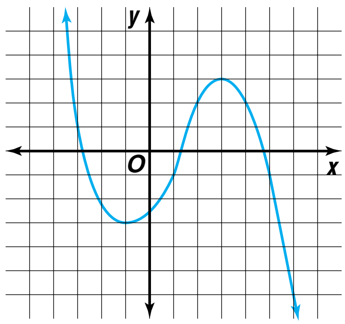
Guided Practice

Locate the extrema for the graph of $y = f(x)$. Name and classify the extrema of the function.

4.



5.



Use a graphing calculator to graph each function and to determine and classify its extrema.

6. $f(x) = 2x^5 - 5x^4$

7. $g(x) = x^4 + 3x^3 - 2$

Determine whether the given critical point is the location of a *maximum*, a *minimum*, or a *point of inflection*.

8. $y = 3x^3 - 9x - 5,\ x = -1$

9. $y = x^2 + 5x - 6,\ x = -2.5$

10. $y = 2x^3 - x^5,\ x = 0$

11. $y = x^6 - 3x^4 + 3x^2 - 1,\ x = 0$

12. **Agriculture** Malik Davis is a soybean farmer. If he harvests his crop now, the yield will average 120 bushels of soybeans per acre and will sell for \$0.48 per bushel. However, he knows that if he waits, his yield will increase by about 10 bushels per week, but the price will decrease by \$0.03 per bushel per week.
    a. If x represents the number of weeks Mr. Davis waits to harvest his crop, write and graph a function $P(x)$ to represent his profit.
    b. How many weeks should Mr. Davis wait in order to maximize his profit?
    c. What is the maximum profit?
    d. What are the risks of waiting?

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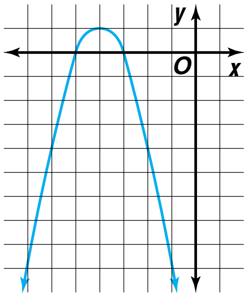
CONTENTS

EXERCISES

Practice

Locate the extrema for the graph of $y = f(x)$. Name and classify the extrema of the function.

13.



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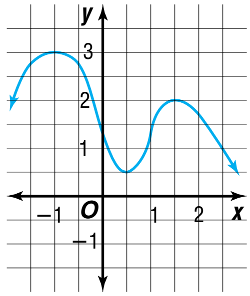

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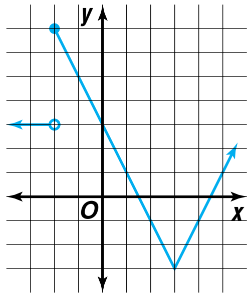

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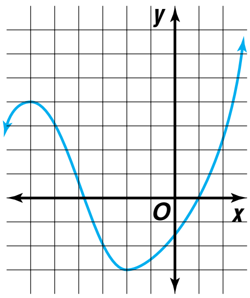

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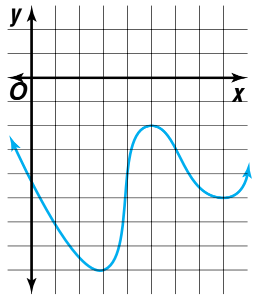

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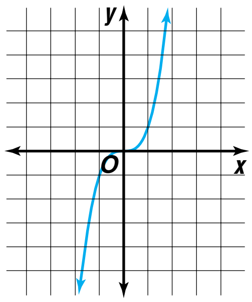


Use a graphing calculator to graph each function and to determine and classify its extrema.

19. $f(x) = -4 + 3x - x^2$

20. $V(w) = w^3 - 7w - 6$

21. $g(x) = 6x^3 + x^2 - 5x - 2$

22. $h(x) = x^4 - 4x^2 - 2$

23. $f(x) = 2x^5 + 4x^2 - 2x - 3$

24. $D(t) = t^3 + t$

25. Determine and classify the extrema of $f(x) = x^4 + 5x^3 + 3x^2 - 4x$.

Determine whether the given critical point is the location of a *maximum,* a *minimum,* or a *point of inflection.*

26. $y = x^3, x = 0$

27. $y = -x^2 + 8x - 10, x = 4$

28. $y = 2x^2 + 10x - 7, x = -2.5$

29. $y = x^4 - 2x^2 + 7, x = 0$

30. $y = \frac{1}{4}x^4 - 2x^2, x = 2$

31. $y = x^3 - 9x^2 + 27x - 27, x = 3$

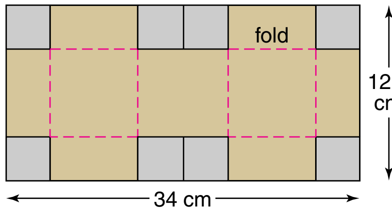
32. $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 1, x = -2$

33. $y = x^3 - x^2 + 3, x = \frac{2}{3}$

34. A function f has a relative maximum at $x = 2$ and a point of inflection at $x = -1$. Find the critical points of $y = -2f(x + 5) - 1$. Describe what happens at each new critical point.

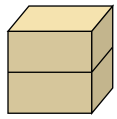
Applications and Problem Solving

35. Manufacturing A 12.5 centimeter by 34 centimeter piece of cardboard will have eight congruent squares removed as in the diagram. The box will be folded to create a take-out hamburger box.



a. Find the model for the volume $V(x)$ of the box as a function of the length x of the sides of the eight squares removed.

b. What are the dimensions of each of the eight squares that should be removed to produce a box with maximum volume?



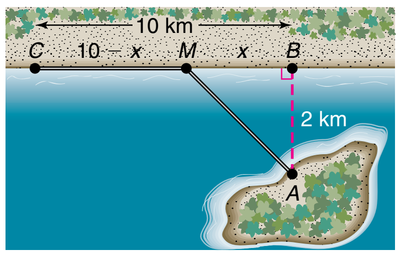
c. Construct a physical model of the box and measure its volume. Compare this result to the result from the mathematical model.

36. Business The Carlisle Innovation Company has created a new product that costs $25 per item to produce. The company has hired a marketing analyst to help it determine a selling price for the product. After collecting and analyzing data relating selling price s to yearly consumer demand d, the analyst estimates demand for the product using the equation $d = -200s + 15{,}000$.

a. If yearly profit is the difference between total revenue and production costs, determine a selling price s, $s \geq 25$, that will maximize the company's yearly profit, P. (*Hint:* $P = sd - 25d$)

b. What are the risks of determining a selling price using this method?

37. Telecommunications A cable company wants to provide service for residents of an island. The distance from the closest point on the island's beach, point A, directly to the mainland at point B is 2 kilometers. The nearest cable station, point C, is 10 kilometers downshore from point B. It costs $3500 per kilometer to lay the cable lines underground and $5000 per kilometer to lay the cable lines under water. The line comes to the mainland at point M. Let x be the distance in kilometers from point B to point M.



a. Write a function to calculate the cost of laying the cable.

b. At what distance x should the cable come to shore to minimize cost?

38. Critical Thinking Which families of graphs have points of inflection but no maximum or minimum points?

39. Physics When the position of a particle as a function of time t is modeled by a polynomial function, then the particle is at rest at each critical point. If a particle has a position given by $s(t) = 2t^3 - 11t^2 + 3t - 9$, find the position of the particle each time it is at rest.

40. **Critical Thinking** A cubic polynomial can have 1 or 3 critical points. Describe the possible combinations of relative maxima and minima for a cubic polynomial.

Mixed Review

41. Is $y = \frac{5x}{x^2 - 3x - 10}$ continuous at $x = 5$? Justify your answer by using the continuity test. *(Lesson 3-5)*

42. Graph the inequality $y \leq \frac{1}{5}(x - 2)^3$. *(Lesson 3-3)*

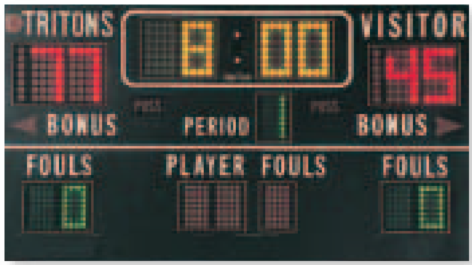
43. **Manufacturing** The Eastern Minnesota Paper Company can convert wood pulp to either newsprint or notebook paper. The mill can produce up to 200 units of paper a day, and regular customers require 10 units of notebook paper and 80 units of newsprint per day. If the profit on a unit of notebook paper is \$400 and the profit on a unit of newsprint is \$350, how much of each should the plant produce? *(Lesson 2-7)*

44. **Geometry** Find the system of inequalities that will define a polygonal convex set that includes all points in the interior of a square whose vertices are $A(-3, 4)$, $B(2, 4)$, $C(2, -1)$, and $D(-3, -1)$. *(Lesson 2-6)*

45. Find the determinant for $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. Does an inverse exist for this matrix? *(Lesson 2-5)*

46. Find $3A + 2B$ if $A = \begin{bmatrix} 4 & -2 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ -4 & 3 \end{bmatrix}$. *(Lesson 2-3)*

47. **Sports** Jon played in two varsity basketball games. He scored 32 points by hitting 17 of his 1-point, 2-point, and 3-point attempts. He made 50% of his 18 2-point field goal attempts. Find the number of 1-point free throws, 2-point field goals, and 3-point field goals Jon scored in these two games. *(Lesson 2-2)*

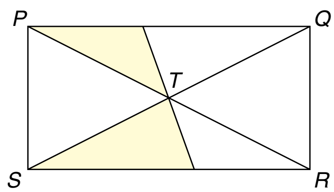


48. Graph $y + 6 \geq 4$. *(Lesson 1-8)*

49. Determine whether the graphs of $2x + 3y = 15$ and $6x = 4y + 16$ are *parallel, coinciding, perpendicular,* or *none of these.* *(Lesson 1-5)*

50. Describe the difference between a relation and a function. How do you test a graph to determine if it is the graph of a function? *(Lesson 1-1)*

51. **SAT/ACT Practice** Refer to the figure at the right. What percent of the area of rectangle *PQRS* is shaded?



A 20%

B $33\frac{1}{3}\%$

C 30%

D 25%

E 40%

Extra Practice See p. A31.

3-7 Graphs of Rational Functions

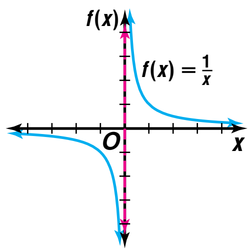
OBJECTIVES
- Graph rational functions.
- Determine vertical, horizontal, and slant asymptotes.

Real World Application

CHEMISTRY The creation of standard chemical solutions requires precise measurements and appropriate mixing. For example, a chemist may have 50 liters of a 10-molar sodium chloride (NaCl) solution (10 moles of NaCl per liter of water.) This solution must be diluted so it can be used in an experiment. Adding 4-molar NaCl solution (4 moles of NaCl per liter of water) to the 10-molar solution will decrease the concentration. The concentration, C, of the mixture can be modeled by $C(x) = \frac{500 + 4x}{50 + x}$, where x is the number of liters of 4-molar solution added. The graph of this function has characteristics that are helpful in understanding the situation. *A problem related to this will be solved in Example 3.*

The concentration function given above is an example of a **rational function.** A rational function is a quotient of two polynomial functions. It has the form $f(x) = \frac{g(x)}{h(x)}$, where $h(x) \neq 0$. The parent rational function is $f(x) = \frac{1}{x}$.

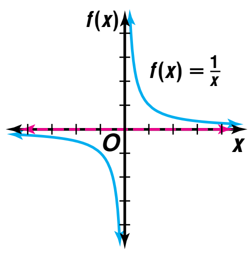
The graph of $f(x) = \frac{1}{x}$ consists of two branches, one in Quadrant I and the other in Quadrant III. The graph has no x- or y-intercepts. The graph of $f(x) = \frac{1}{x}$, like that of many rational functions, has branches that approach lines called **asymptotes.** In the case of $f(x) = \frac{1}{x}$, the line $x = 0$ is a **vertical asymptote.** Notice that as x approaches 0 from the right, the value of $f(x)$ increases without bound toward positive infinity (∞). As x approaches 0 from the left, the value of $f(x)$ decreases without bound toward negative infinity ($-\infty$).



A vertical asymptote in the graph of a rational function is also called a pole of the function.

Vertical Asymptote	The line $x = a$ is a vertical asymptote for a function $f(x)$ if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$ from either the left or the right.

Also notice for $f(x) = \frac{1}{x}$ that as the value of x increases and approaches positive infinity, the value of $f(x)$ approaches 0. The same type of behavior can also be observed in the third quadrant: as x decreases and approaches negative infinity, the value of $f(x)$ approaches 0. When the values of a function approach a constant value b as $x \to \infty$ or $x \to -\infty$, then the function has a **horizontal asymptote**. Thus, for $f(x) = \frac{1}{x}$, the line $f(x) = 0$ is a horizontal asymptote.



Horizontal Asymptote	The line $y = b$ is a horizontal asymptote for a function $f(x)$ if $f(x) \to b$ as $x \to \infty$ or as $x \to -\infty$.

There are several methods that may be used to determine if a rational function has a horizontal asymptote. Two such methods are used in the example below.

Example 1 **Determine the asymptotes for the graph of $f(x) = \dfrac{3x - 1}{x - 2}$.**

Since $f(2)$ is undefined, there may be a vertical asymptote at $x = 2$. To verify that $x = 2$ is indeed a vertical asymptote, you have to make sure that $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to 2$ from either the left or the right.

x	$f(x)$
1.9	−47
1.99	−497
1.999	−4997
1.9999	−49997

The values in the table suggest that $f(x) \to -\infty$ as $x \to 2$ from the left, so there is a vertical asymptote at $x = 2$.

Two different methods may be used to find the horizontal asymptote.

Method 1

Let $f(x) = y$ and solve for x in terms of y. Then find where the function is undefined for values of y.

$$y = \frac{3x - 1}{x - 2}$$

$$y(x - 2) = 3x - 1 \quad \textit{Multiply each side by } (x - 2).$$

$$xy - 2y = 3x - 1 \quad \textit{Distribute.}$$

$$xy - 3x = 2y - 1$$

$$x(y - 3) = 2y - 1 \quad \textit{Factor.}$$

$$x = \frac{2y - 1}{y - 3} \quad \textit{Divide each side by } y - 3.$$

The rational expression $\dfrac{2y - 1}{y - 3}$ is undefined for $y = 3$. Thus, the horizontal asymptote is the line $y = 3$.

Method 2

First divide the numerator and denominator by the highest power of x.

$$y = \frac{3x - 1}{x - 2}$$

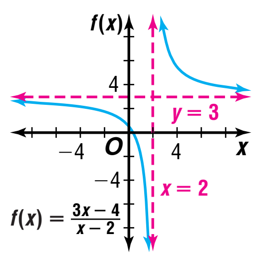
$$y = \frac{\frac{3x}{x} - \frac{1}{x}}{\frac{x}{x} - \frac{2}{x}}$$

$$y = \frac{3 - \frac{1}{x}}{1 - \frac{2}{x}}$$

As the value of x increases positively or negatively, the values of $\frac{1}{x}$ and $\frac{2}{x}$ approach zero. Therefore, the value of the entire expression approaches $\frac{3}{1}$ or 3. So, the line $y = 3$ is the horizontal asymptote.

Method 2 is preferable when the degree of the numerator is greater than that of the denominator.

The graph of this function verifies that the lines $y = 3$ and $x = 2$ are asymptotes.



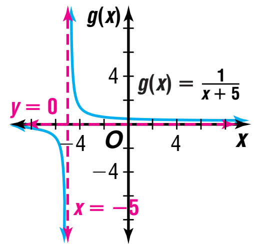
Graphing Calculator Tip

You can use the TRACE or TABLE function on your graphing calculator to approximate the vertical or horizontal asymptote.

Example 2 **Use the parent graph $f(x) = \frac{1}{x}$ to graph each function. Describe the transformation(s) that take place. Identify the new location of each asymptote.**

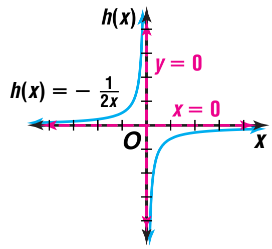
a. $g(x) = \frac{1}{x + 5}$

To graph $g(x) = \frac{1}{x + 5}$, translate the parent graph 5 units to the left. The new vertical asymptote is $x = -5$. The horizontal asymptote, $y = 0$, remains unchanged.



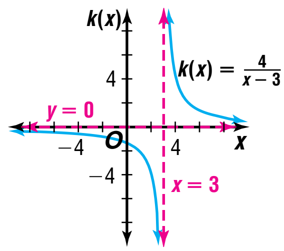
b. $h(x) = -\frac{1}{2x}$

To graph $h(x) = -\frac{1}{2x}$, reflect the parent graph over the x-axis, and compress the result horizontally by a factor of 2. This does not affect the vertical asymptote at $x = 0$. The horizontal asymptote, $y = 0$, is also unchanged.



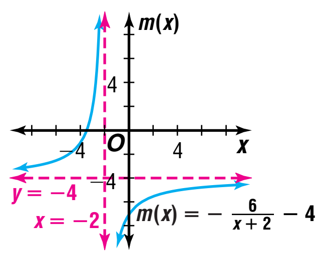
c. $k(x) = \frac{4}{x - 3}$

To graph $k(x) = \frac{4}{x - 3}$, stretch the parent graph vertically by a factor of 4 and translate the parent graph 3 units to the right. The new vertical asymptote is $x = 3$. The horizontal asymptote, $y = 0$, is unchanged.



d. $m(x) = -\frac{6}{x + 2} - 4$

To graph $m(x) = -\frac{6}{x + 2} - 4$, reflect the parent graph over the x-axis, stretch the result vertically by a factor of 6, and translate the result 2 units to the left and 4 units down. The new vertical asymptote is $x = -2$. The horizontal asymptote changes from $y = 0$ to $y = -4$.



Many real-world situations can be modeled by rational functions.

Example **3** **CHEMISTRY** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Write the function as a transformation of the parent $f(x) = \frac{1}{x}$.

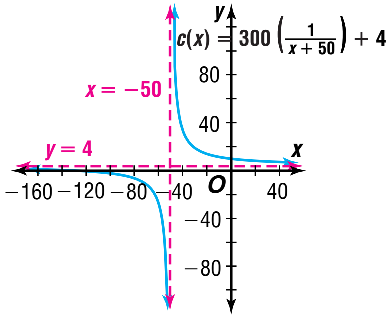
b. Graph the function and identify the asymptotes. Interpret their meaning in terms of the problem.

a. In order to write $C(x) = \frac{500 + 4x}{50 + x}$ as a transformation of $f(x) = \frac{1}{x}$, you must perform the indicated division of the linear functions in the numerator and denominator.

$$\begin{array}{r} 4 \\ x + 50\overline{)4x + 500} \\ \underline{4x + 200} \\ 300 \end{array} \quad \Rightarrow \quad 4 + \frac{300}{x + 50} \text{ or } 300\left(\frac{1}{x + 50}\right) + 4$$

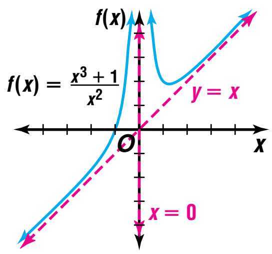
Therefore, $C(x) = 300\left(\frac{1}{x + 50}\right) + 4$.

b. The graph of $C(x)$ is a vertical stretch of the parent function by a factor of 300 and a translation 50 units to the left and 4 units up. The asymptotes are $x = -50$ and $y = 4$.



For the purposes of our model, the domain of $C(x)$ must be restricted to $x \geq 0$ since the number of liters of the 4-molar solution added cannot be negative. The vertical asymptote $x = -50$ has no meaning in this application as a result of this restriction. The horizontal asymptote indicates that as the amount of 4-molar solution that has been added grows, the overall concentration will approach a molarity of 4.

A third type of asymptote is a **slant asymptote.** Slant asymptotes occur when the degree of the numerator of a rational function is *exactly* one greater than that of the denominator. For example, in $f(x) = \frac{x^3 + 1}{x^2}$, the degree of the numerator is 3 and the degree of the denominator is 2. Therefore, this function has a slant asymptote, as shown in the graph. Note that there is also a vertical asymptote at $x = 0$. *When the degrees are the same or the denominator has the greater degree, the function has a horizontal asymptote.*



Slant Asymptote	The oblique line ℓ is a slant asymptote for a function $f(x)$ if the graph of $y = f(x)$ approaches ℓ as $x \to \infty$ or as $x \to -\infty$.

Example 4 **Determine the slant asymptote for $f(x) = \frac{2x^2 - 3x + 1}{x - 2}$.**

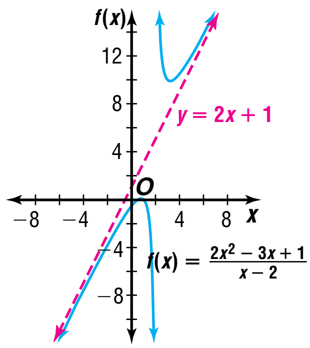
This method can also be used to determine the horizontal asymptote when the degree of the numerator is equal to or greater than that of the denominator.

First, use division to rewrite the function.

$$\begin{array}{r} 2x + 1 \\ x - 2\overline{)2x^2 - 3x + 1} \\ \underline{2x^2 - 4x} \\ x + 1 \\ \underline{x - 2} \\ 3 \end{array} \quad \Rightarrow \quad f(x) = 2x + 1 + \frac{3}{x - 2}$$

As $x \to \infty$, $\frac{3}{x - 2} \to 0$. So, the graph of $f(x)$ will approach that of $y = 2x + 1$. This means that the line $y = 2x + 1$ is a slant asymptote for the graph of $f(x)$. *Note that $x = 2$ is a vertical asymptote.*

The graph of this function verifies that the line $y = 2x + 1$ is a slant asymptote.



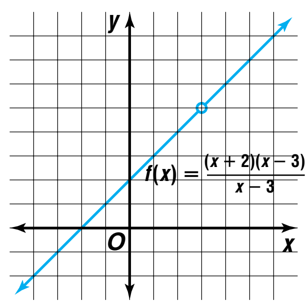
There are times when the numerator and denominator of a rational function share a common factor. Consider $f(x) = \frac{(x + 2)(x - 3)}{x - 3}$. Since an x-value of 3 results in a denominator of 0, you might expect there to be a vertical asymptote at $x = 3$. However, $x - 3$ is a common factor of the numerator and denominator.

Numerically, we can see that $f(x)$ has the same values as $g(x) = x + 2$ except at $x = 3$.

x	2.9	2.99	3.0	3.01	3.1
f(x)	4.9	4.99	—	5.01	5.1
g(x)	4.9	4.99	5.0	5.01	5.1

The y-values on the graph of f approach 5 from both sides but never get to 5, so the graph has point discontinuity at (3, 5).

Whenever the numerator and denominator of a rational function contain a common linear factor, a point discontinuity may appear in the graph of the function. If, after dividing the common linear factors, the same factor remains in the denominator, a vertical asymptote exists. Otherwise, the graph will have point discontinuity.

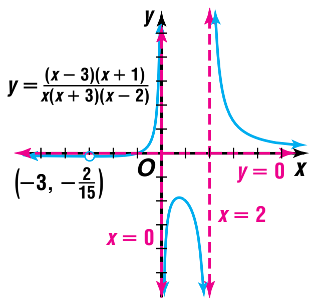


Example 5 **Graph $y = \dfrac{(x+3)(x+1)}{x(x+3)(x-2)}$.**

$$y = \frac{(x+3)(x+1)}{x(x+3)(x-2)}$$
$$= \frac{x+1}{x(x-2)},\ x \neq -3$$

Since $x + 3$ is a common factor that does not remain in the denominator after the division, there is point discontinuity at $x = -3$. Because y increases or decreases without bound close to $x = 0$ and $x = 2$, there are vertical asymptotes at $x = 0$ and $x = 2$. There is also a horizontal asymptote at $y = 0$.

The graph is the graph of $y = \dfrac{x+1}{x(x-2)}$, except for point discontinuity at $\left(-3, -\dfrac{2}{15}\right)$.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Graph** $f(x) = \dfrac{1}{x}$ after it has been translated 2 units to the right and down 6 units.
   a. What are its asymptotes?
   b. Write an equation of the translated graph.
2. **Draw** a graph to illustrate each type of asymptote discussed in this lesson: vertical, horizontal, and slant.
3. **Write** an equation of a rational function that has point discontinuity.
4. **True or False:** If a value of x causes a zero in the denominator of a rational function, then there is a vertical asymptote at that x-value. Explain.

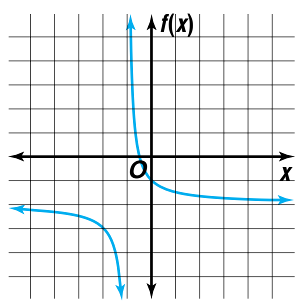
Guided Practice

Determine the equations of the vertical and horizontal asymptotes, if any, of each function.

5. $f(x) = \dfrac{x}{x-5}$

6. $g(x) = \dfrac{x^3}{(x-2)(x+1)}$

7. The graph at the right shows a transformation of $f(x) = \dfrac{1}{x}$. Write an equation of the function.



Use the parent graph $f(x) = \frac{1}{x}$ to graph each equation. Describe the transformation(s) that have taken place. Identify the new locations of the asymptotes.

8. $y = \frac{1}{x - 4}$

9. $y = \frac{1}{x + 2} - 1$

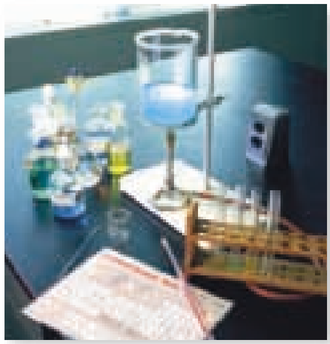
10. Determine the slant asymptote for the function $f(x) = \frac{3x^2 - 4x + 5}{x - 3}$.

Graph each function.

11. $y = \frac{x + 2}{(x + 1)(x - 1)}$

12. $y = \frac{x^2 + 4x + 4}{x + 2}$

13. Chemistry The Ideal Gas Law states that the pressure P, volume V, and temperature T, of an ideal gas are related by the equation $PV = nRT$, where n is the number of moles of gas and R is a constant.



a. Sketch a graph of P versus V, assuming that T is fixed.

b. What are the asymptotes of the graph?

c. What happens to the pressure of the gas if the temperature is held fixed and the gas is allowed to occupy a larger and larger volume?

EXERCISES

Practice

Determine the equations of the vertical and horizontal asymptotes, if any, of each function.

14. $f(x) = \frac{2x}{x + 4}$

15. $f(x) = \frac{x^2}{x + 6}$

16. $g(x) = \frac{x - 1}{(2x + 1)(x - 5)}$

17. $g(x) = \frac{x - 2}{x^2 + 4x + 3}$

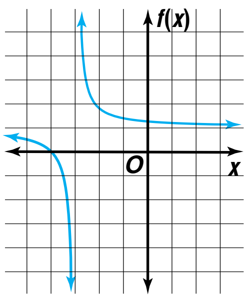
18. $h(x) = \frac{x^2}{x^2 + 1}$

19. $h(x) = \frac{(x + 1)^2}{x^2 - 1}$

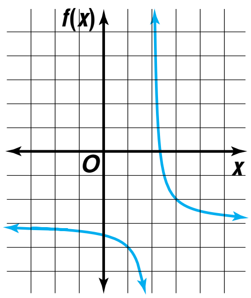
20. What are the vertical and horizontal asymptotes of the function $y = \frac{x^3}{(x - 2)^4}$?

Each graph below shows a transformation of $f(x) = \frac{1}{x}$. Write an equation of each function.

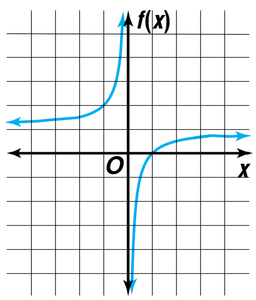
21.



22.



23.



Use the parent graph $f(x) = \frac{1}{x}$ to graph each equation. Describe the transformation(s) that have taken place. Identify the new locations of the asymptotes.

24. $y = \frac{1}{x} + 3$ **25.** $y = \frac{2}{x-4}$ **26.** $y = \frac{2}{x+3} - 1$

27. $y = -\frac{3}{x} + 2$ **28.** $y = \frac{3x+1}{x-3}$ **29.** $y = \frac{-4x+2}{x+5}$

Determine the slant asymptote of each function.

30. $f(x) = \frac{x^2+3x-3}{x+4}$ **31.** $f(x) = \frac{x^2+3x-4}{x}$

32. $f(x) = \frac{x^3-2x^2+x-4}{x^2+1}$ **33.** $f(x) = \frac{x^2-4x+1}{2x-3}$

34. Does the function $f(x) = \frac{x^3-4x^2+2x-6}{x+3}$ have a slant asymptote? If so, find an equation of the slant asymptote. If not, explain.

Graph each function.

35. $y = \frac{(x-2)(x+1)}{x}$ **36.** $y = \frac{x^2-4}{x-2}$ **37.** $y = \frac{x+2}{x^2-4}$

38. $y = \frac{(x-2)^2(x+1)^2}{(x-2)(x-1)}$ **39.** $y = \frac{x^2-6x+9}{x^2-x-6}$ **40.** $y = \frac{x^2-1}{x^2-2x+1}$

Applications and Problem Solving

41. Chemistry Suppose the chemist in the application at the beginning of the lesson had to dilute 40 liters of a 12-molar solution by adding 3-molar solution.

a. Write the function that models the concentration of the mixture as a function of the number of liters t of 3-molar solution added.

b. How many liters of 3-molar solution must be added to create a 10-molar solution?

42. Electronics Suppose the current I in an electric circuit is given by the formula $I = t + \frac{1}{10-t}$, where t is time. What happens to the circuit as t approaches 10?

43. Critical Thinking Write an equation of a rational function whose graph has all of the following characteristics.

- x-intercepts at $x = 2$ and $x = -3$;
- a vertical asymptote at $x = 4$; and
- point discontinuity at $(-5, 0)$.

44. Geometry The volume of a rectangular prism with a square base is fixed at 120 cubic feet.

a. Write the surface area of the prism as a function $A(x)$ of the length of the side of the square x.

b. Graph the surface area function.

c. What happens to the surface area of the prism as the length of the side of the square approaches 0?

45. Critical Thinking The graph of a rational function cannot intersect a vertical asymptote, but it is possible for the graph to intersect its horizontal asymptote for small values of x. Give an example of such a rational function. (*Hint:* Let the x-axis be the horizontal asymptote of the function.)

46. **Physics** Like charges repel and unlike charges attract. Coulomb's Law states that the force F of attraction or repulsion between two charges, q_1 and q_2, is given by $F = \frac{kq_1q_2}{r^2}$, where k is a constant and r is the distance between the charges. Suppose you were to graph F as a function of r for two positive charges.

a. What asymptotes would the graph have?

b. Interpret the meaning of the asymptotes in terms of the problem.

47. **Analytic Geometry** Recall from Exercise 36 of Lesson 3-5 that a *secant* is a line that intersects the graph of a function in two or more points. Consider the function $f(x) = x^2$.

a. Find an expression for the slope of the secant through the points at (3, 9) and (a, a^2).

b. What happens to the slope of the secant as a approaches 3?

Mixed Review

48. Find and classify the extrema of the function $f(x) = -x^2 + 4x - 3$. *(Lesson 3-6)*

49. Find the inverse of $x^2 - 9 = y$. *(Lesson 3-4)*

50. Find the maximum and minimum values of $f(x, y) = y - x$ defined for the polygonal convex set having vertices at (0, 0), (4, 0), (3, 5), and (0, 5). *(Lesson 2-6)*

51. Find $-4\begin{bmatrix} -6 & 5 \\ 8 & -4 \end{bmatrix}$. *(Lesson 2-3)*

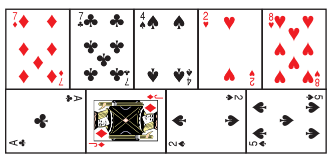
52. **Consumer Awareness** Bill and Liz are going on a vacation in Jamaica. Bill bought 8 rolls of film and 2 bottles of sunscreen for \$35.10. The next day, Liz paid \$14.30 for three rolls of film and one bottle of sunscreen. If the price of each bottle of sunscreen is the same and the price of each roll of film is the same, what is the price of a roll of film and a bottle of sunscreen? *(Lesson 2-1)*

53. Of (0, 0), (3, 2), (−4, 2), or (−2, 4), which satisfy $x + y \geq 3$? *(Lesson 1-8)*

54. Write the equation $15y - x = 1$ in slope-intercept form. *(Lesson 1-4)*

55. Find $[f \circ g](x)$ and $[g \circ f](x)$ for $f(x) = 8x$ and $g(x) = 2 - x^2$. *(Lesson 1-2)*

56. **SAT/ACT Practice** Nine playing cards from the same deck are placed to form a large rectangle whose area is 180 square inches. There is no space between the cards and no overlap. What is the perimeter of this rectangle?



A 29 in. B 58 in. C 64 in.
D 116 in. E 210 in.

Extra Practice See p. A31.

CONTENTS

3-8

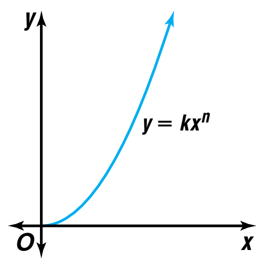
Direct, Inverse, and Joint Variation

OBJECTIVE
- Solve problems involving direct, inverse, and joint variation.

AUTO SAFETY The faster you drive, the more time and distance you need to stop safely, and the less time you have to react. The stopping distance for a vehicle is the sum of the reaction distance d_1, how far the car travels before you hit the brakes, and the braking distance d_2, how far the car travels once the brakes are applied. According to the National Highway Traffic Safety Administration (NHTSA), under the best conditions the reaction time t of most drivers is about 1.5 seconds. Once brakes are applied, the braking distance varies directly as the square of the car's speed s. *A problem related to this will be solved in Example 2.*

The relationship between braking distance and car speed is an example of a **direct variation.** As the speed of the car increases, the braking distance also increases at a constant rate.

A direct variation can be described by the equation $y = kx^n$. The k in this equation is called the **constant of variation.** To express a direct variation, we say that *y varies directly as x^n.*

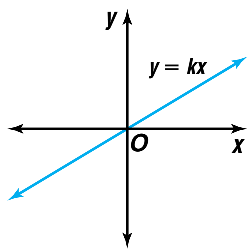


Direct Variation

y varies directly as x^n if there is some nonzero constant k such that $y = kx^n$, $n > 0$. k is called the *constant of variation.*

Notice when $n = 1$, the equation of direct variation simplifies to $y = kx$. This is the equation of a line through the origin written in slope-intercept form. In this special case, the constant of variation is the slope of the line.

To find the constant of variation, substitute known values of x^n and y in the equation $y = kx^n$ and solve for k.



Example 1 **Suppose y varies directly as x and $y = 27$ when $x = 6$.**

a. Find the constant of variation and write an equation of the form $y = kx^n$.

b. Use the equation to find the value of y when $x = 10$.

a. In this case, the power of x is 1, so the direct variation equation is $y = kx$.

$$y = kx$$
$$27 = k(6) \quad y = 27, x = 6$$
$$\frac{27}{6} = k \quad \text{Divide each side by 6.}$$
$$4.5 = k$$

The constant of variation is 4.5. The equation relating x and y is $y = 4.5x$.

b. $y = 4.5x$
$y = 4.5(10)$ *$x = 10$*
$y = 45$

When $x = 10$, the value of y is 45.

Direct variation equations are used frequently to solve real-world problems.

Example 2 Real World Application

AUTO SAFETY **Refer to the application at the beginning of this lesson. The NHTSA reports an average braking distance of 227 feet for a car traveling 60 miles per hour, with a total stopping distance of 359 feet.**

a. Write an equation of direct variation relating braking distance to car speed. Then graph the equation.

b. Use the equation to find the braking distance required for a car traveling 70 miles per hour.

c. Calculate the total stopping distance required for a car traveling 70 miles per hour.

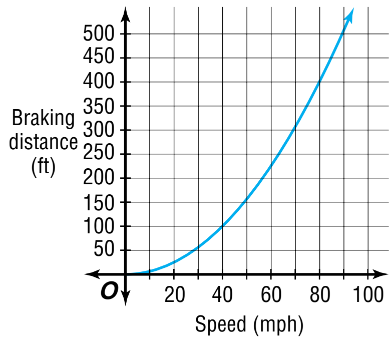
a. First, translate the statement of variation into an equation of the form $y = kx^n$.

The braking distance varies directly as the square of the car's speed.

$$d_2 = k \cdot s^2$$

Then, substitute corresponding values for braking distance and speed in the equation and solve for k.

$d_2 = ks^2$
$227 = k(60)^2$ *$d_2 = 227,\ s = 60$*
$\dfrac{227}{(60)^2} = k$ *Divide each side by $(60)^2$.*
$0.063 \approx k$



The constant of variation is approximately 0.063. Thus, the equation relating braking distance to car speed is $d_2 = 0.063s^2$. Notice the graph of the variation equation is a parabola centered at the origin and opening up.

b. Evaluate the equation of variation for $s = 70$.

$d_2 = 0.063s^2$
$d_2 = 0.063(70)^2$ *$s = 70$*
$d_2 = 308.7$ or about 309

The braking distance for a car traveling 70 mph is about 309 feet.

c. To find the total stopping distance, first calculate the reaction distance given a reaction time of 1.5 seconds and a speed of 70 miles per hour. Since time is in seconds, we need to change miles per hour to feet per second.

$$\frac{70\ \cancel{\text{mi}}}{1\ \cancel{\text{h}}} \cdot \frac{1\ \cancel{\text{h}}}{60\ \cancel{\text{min}}} \cdot \frac{1\ \cancel{\text{min}}}{60\ \text{s}} \cdot \frac{5280\ \text{ft}}{1\ \cancel{\text{mi}}} \approx 103\ \text{ft/s}$$

$d_1 = st$ *reaction distance = rate of speed · time*

$d_1 = (103)(1.5)$ or about 155 feet *$s \approx 103$ ft/s; $t = 1.5$ s*

The total stopping distance is then the sum of the reaction and braking distances, $155 + 309$ or 464 feet.

If you know that y varies directly as x^n and one set of values, you can use a proportion to find the other set of corresponding values.

Suppose $y_1 = kx_1{}^n$ and $y_2 = kx_2{}^n$. Solve each equation for k.

$$\frac{y_1}{x_1} = k \text{ and } \frac{y_2}{x_2} = k \Rightarrow \frac{y_1}{x_1} = \frac{y_2}{x_2}$$ *Since both ratios equal k, the ratios are equal.*

Using the properties of proportions, you can find many other proportions that relate these same x^n- and y-values. *You will derive another of these proportions in Exercise 2.*

Example 3 **If y varies directly as the cube of x and $y = -67.5$ when $x = 3$, find x when $y = -540$.**

Use a proportion that relates the values.

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}$$

$$\frac{-67.5}{(3)^3} = \frac{-540}{(x_2)^3}$$ *Substitute the known values.*

$$-67.5(x_2)^3 = -540(3)^3$$ *Cross multiply.*

$$(x_2)^3 = 216$$

$$x_2 = \sqrt[3]{216} \text{ or } 6$$ *Take the cube root of each side.*

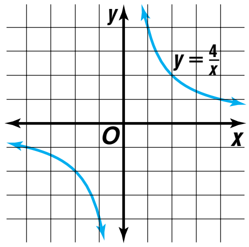
When $y = -540$, the value of x is 6. *This is a reasonable answer for x, since as y decreased, the value of x increased.*

Many quantities are **inversely proportional** or are said to *vary inversely* with each other. This means that as one value increases the other decreases and vice versa. For example, elevation and air temperature vary inversely with each other. When you travel to a higher elevation above Earth's surface, the air temperature decreases.

Inverse Variation

y varies inversely as x^n if there is some nonzero constant k such that $x^n y = k$ or $y = \frac{k}{x^n}$, $n > 0$.

Suppose y varies inversely as x such that $xy = 4$ or $y = \frac{4}{x}$. The graph of this equation is shown at the right. Notice that in this case, k is a positive value, 4, so as the value of x increases, the value of y decreases.



Just as with direct variation, a proportion can be used with inverse variation to solve problems where some quantities are known. The proportion shown below is only one of several that can be formed.

$$x_1^n y_1 = k \text{ and } x_2^n y_2 = k$$

$$x_1^n y_1 = x_2^n y_2 \qquad \textit{Substitution property of equality}$$

$$\frac{x_1^n}{y_2} = \frac{x_2^n}{y_1} \qquad \textit{Divide each side by } y_1y_2.$$

Other possible proportions include $\frac{y_1}{y_2} = \frac{x_2^n}{x_1^n}$, $\frac{y_2}{x_1^n} = \frac{y_1}{x_2^n}$, and $\frac{x_2^n}{x_1^n} = \frac{y_1}{y_2}$.

Example 4 **If y varies inversely as x and $y = 21$ when $x = 15$, find x when $y = 12$.**

Use a proportion that relates the values.

$$\frac{x_1}{y_2} = \frac{x_2}{y_1} \qquad n = 1$$

$$\frac{15}{12} = \frac{x_2}{21} \qquad \textit{Substitute the known values.}$$

$$12x_2 = 315 \qquad \textit{Cross multiply.}$$

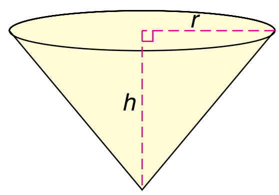
$$x_2 = \frac{315}{12} \text{ or } 26.25 \qquad \textit{Divide each side by 15.}$$

When $y = 12$, the value of x is 26.25.

Another type of variation is **joint variation.** This type of variation occurs when one quantity varies directly as the product of two or more other quantities.

Joint Variation	y varies jointly as x^n and z^n if there is some nonzero constant k such that $y = kx^nz^n$, where $x \neq 0$, $z \neq 0$, and $n > 0$.

Example 5 **GEOMETRY** **The volume V of a cone varies jointly as the height h and the square of the radius r of the base. Find the equation for the volume of a cone with height 6 centimeters and base diameter 10 centimeters that has a volume of 50π cubic centimeters.**



Read the problem and use the known values of V, h, and r to find the equation of joint variation.

$$\underbrace{\text{The volume}}_{V}\ \underbrace{\text{varies jointly as}}_{=\ k}\ \underbrace{\text{the height}}_{\cdot\ h}\ \underbrace{\text{and}}_{\cdot}\ \underbrace{\text{the square of the radius.}}_{r^2}$$

$$
\begin{aligned}
V &= khr^2 \\
50\pi &= k(6)(5)^2 && V = 50\pi,\ h = 6,\ r = 5 \\
50\pi &= 150k \\
\frac{\pi}{3} &= k && \text{Solve for } k.
\end{aligned}
$$

The equation for the volume of a cone is $V = \frac{\pi}{3}hr^2$.

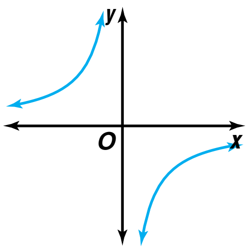
CHECK FOR UNDERSTANDING

Communicating Mathematics

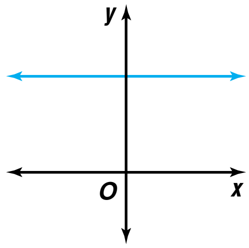
Read and study the lesson to answer each question.

1. **Describe** each graph below as illustrating a direct variation, inverse variation, or neither.

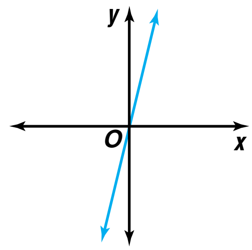
a.



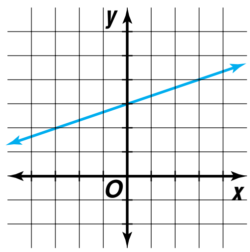
b.



c.



2. **Derive** a different proportion from that used in Example 3 relating x^n and y if y varies directly as x.
3. **Explain** why the graph at the right does not represent a direct variation.



4. **Write** statements relating two quantities in real life that exemplify each type of variation.
   a. direct   b. inverse   c. joint

Guided Practice

Find the constant of variation for each relation and use it to write an equation for each statement. Then solve the equation.

5. If y varies inversely as x and $y = 3$ when $x = 4$, find y when $x = 15$.
6. If y varies directly as the square of x and $y = -54$ when $x = 9$, find y when $x = 6$.
7. If y varies jointly as x and the cube of z and $y = 16$ when $x = 4$ and $z = 2$, find y when $x = -8$ and $z = -3$.
8. If y varies jointly as x and z and inversely as the square of w, and $y = 3$ when $x = 3$, $z = 10$, and $w = 2$, find y when $x = 4$, $z = 20$, and $w = 4$.

Write a statement of variation relating the variables of each equation. Then name the constant of variation.

9. $\frac{x^4}{y} = 7$ 10. $A = \ell w$ 11. $x = \frac{-3}{y}$

12. **Forestry** A lumber company needs to estimate the volume of wood a load of timber will produce. The supervisor knows that the volume of wood in a tree varies jointly as the height h and the square of the tree's girth g. The supervisor observes that a tree 40 meters tall with a girth of 1.5 meters produces 288 cubic meters of wood.

   a. Write an equation that represents this situation.

   b. What volume of wood can the supervisor expect to obtain from 50 trees averaging 75 meters in height and 2 meters in girth?

EXERCISES

Practice

Find the constant of variation for each relation and use it to write an equation for each statement. Then solve the equation.

13. If y varies directly as x and $y = 0.3$ when $x = 1.5$, find y when $x = 6$.

14. If y varies inversely as x and $y = -2$ when $x = 25$, find x when $y = -40$.

15. Suppose y varies jointly as x and z and $y = 36$ when $x = 1.2$ and $z = 2$. Find y when $x = 0.4$ and $z = 3$.

16. If y varies inversely as the square of x and $y = 9$ when $x = 2$, find y when $x = 3$.

17. If r varies directly as the square of t and $r = 4$ when $t = \frac{1}{2}$, find r when $t = \frac{1}{4}$.

18. Suppose y varies inversely as the square root of x and $x = 1.21$ when $y = 0.44$. Find y when $x = 0.16$.

19. If y varies jointly as the cube of x and the square of z and $y = -9$ when $x = -3$ and $z = 2$, find y when $x = -4$ and $z = -3$.

20. If y varies directly as x and inversely as the square of z and $y = \frac{1}{6}$ when $x = 20$ and $z = 6$, find y when $x = 14$ and $z = 5$.

21. Suppose y varies jointly as x and z and inversely as w and $y = -3$ when $x = 2$, $z = -3$, and $w = 4$. Find y when $x = 4$, $z = -7$ and $w = -4$.

22. If y varies inversely as the cube of x and directly as the square of z and $y = -6$ when $x = 3$ and $z = 9$, find y when $x = 6$ and $z = -4$.

23. If a varies directly as the square of b and inversely as c and $a = 45$ when $b = 6$ and $c = 12$, find b when $a = 96$ and $c = 10$.

24. If y varies inversely as the square of x and $y = 2$ when $x = 4$, find x when $y = 8$.

Write a statement of variation relating the variables of each equation. Then name the constant of variation.

25. $C = \pi d$	26. $\frac{x}{y} = 4$	27. $xz^2 = \frac{3}{4}y$
28. $V = \frac{4}{3}\pi r^3$	29. $4x^2 = \frac{5}{y}$	30. $y = \frac{2}{\sqrt{x}}$
31. $A = 0.5h(b_1 + b_2)$	32. $y = \frac{x}{3z^2}$	33. $\frac{1}{7}y = \frac{x^2}{z^3}$

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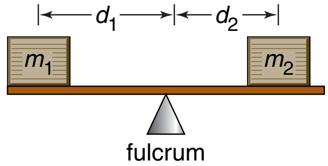
CONTENTS

34. Write a statement of variation for the equation $y = \frac{kx^3 \cdot z}{w^2}$ if k is the constant of variation.

Applications and Problem Solving

Real World Application

35. **Physics** If you have observed people on a seesaw, you may have noticed that the heavier person must sit closer to the fulcrum for the seesaw to balance. In doing so, the heavier participant creates a rotational force, called *torque*. The torque on the end of a seesaw depends on the mass of the person and his or her distance from the seesaw's fulcrum. In order to reduce torque, one must either reduce the distance between the person and the fulcrum or replace the person with someone having a smaller mass.



a. What type of variation describes the relationship between torque, mass, and distance? Explain.

b. Let m_1, d_1, and T_1 be the mass, distance, and torque on one side of a seesaw, and let m_2, d_2, and T_2 be the mass, distance, and torque on the other side. Derive an equation that represents this seesaw in balance.

c. A 75-pound child and a 125-pound babysitter sit at either end of a seesaw. If the child sits 3.3 meters from the fulcrum, use the equation found in part **b** to determine how far the babysitter should sit from the pivot in order to balance the seesaw.

36. **Pool Maintenance** Kai decides to empty her pool for the winter. She knows that the time t required to empty a pool varies inversely as the rate r of pumping.

a. Write an equation that represents this situation. Let k be the constant of variation.

b. In the past, Kai was able to empty her pool in 45 minutes at a rate of 800 liters per minute. She now owns a new pump that can empty the pool at a rate of 1 kiloliter per minute. How long will it take Kai to empty the pool using this new pump?

37. **Critical Thinking** Show that if y varies directly as x, then x varies directly as y.

38. **Movies** The intensity of light I, measured in *lux,* is inversely proportional to the square of the distance d between the light source and the object illuminated.

a. Write an equation that represents this situation.

b. Using a light meter, a lighting director measures the intensity of the light from a bulb hanging 6 feet overhead a circular table at 16 lux. If the table has a 5-foot diameter, what illumination reading will the director find at the edge of the table where the actors will sit? Round to the nearest tenth.

39. **Critical Thinking** If a varies directly as the square of b and inversely as the cube of c, how is the value of a changed when the values of b and c are halved? Explain.

40. **Space Science** Newton's Law of Universal Gravitation states that two objects attract one another with a force F that varies directly as the product of their masses, m_1 and m_2, and inversely as the square of the distance d between their centers. It is this force, called gravity, which pulls objects to Earth and keeps Earth in its orbit about the sun.

a. Write an equation that represents this situation. Let G be the constant of variation.

b. Use the chart at the right to determine the value of G if the force of attraction between Earth and the moon is 1.99×10^{20} newtons (N) and the distance between them is 3.84×10^{8} meters. Be sure to include appropriate units with your answer.

	Mass
Earth	5.98×10^{24} kg
Sun	1.99×10^{30} kg
Moon	7.36×10^{22} kg

c. Using your answers to parts **a** and **b**, find the force of attraction between Earth and the sun if the distance between them is 1.50×10^{11} meters.

d. How many times greater is the force of attraction between the sun and Earth than between the moon and Earth?

41. **Electricity** Wires used to connect electric devices have very small resistances, allowing them to conduct currently more readily. The resistance R of a piece of wire varies directly as its length L and inversely as its cross-sectional area A. A piece of copper wire 2 meters long and 2 millimeters in diameter has a resistance of 1.07×10^{-2} ohms (Ω). Find the resistance of a second piece of copper wire 3 meters long and 6 millimeters in diameter. (*Hint*: The cross-sectional area will be πr^2.)

Mixed Review

42. Graph $y = \frac{x-1}{x^2-9}$. *(Lesson 3-7)*

43. Find the inverse of $f(x) = (x-3)^3 + 6$. Then state whether the inverse is a function. *(Lesson 3-4)*

44. Square $ABCD$ has vertices $A(1, 2)$, $B(3, -2)$, $C(-1, -4)$, and $D(-3, 0)$. Find the image of the square after a reflection over the y-axis. *(Lesson 2-4)*

45. State whether the system $4x - 2y = 7$ and $-12x + 6y = -21$ is *consistent and independent, consistent and dependent,* or *inconsistent.* *(Lesson 2-1)*

46. Graph $g(x) = \begin{cases} 2 \text{ if } x < 1 \\ -x + 5 \text{ if } x \geq 1 \end{cases}$. *(Lesson 1-7)*

47. **Education** In 1995, 23.2% of the student body at Kennedy High School were seniors. By 2000, seniors made up only 18.6% of the student body. Assuming the level of decline continues at the same rate, write a linear equation in slope-intercept form to describe the percent of seniors y in the student body in year x. *(Lesson 1-4)*

48. **SAT/ACT Practice** If $a^2b = 12^2$, and b is an odd integer, then a could be divisible by all of the following EXCEPT

A 3 **B** 4 **C** 6 **D** 9 **E** 12

Extra Practice See p. A31.

VOCABULARY

absolute maximum (p.171)
absolute minimum (p. 171)
asymptotes (p. 180)
constant function (pp. 137, 164)
constant of variation (p. 189)
continuous (p. 160)
critical point (p. 171)
decreasing function (p. 164)
direct variation (p. 189)
discontinuous (p. 159)
end behavior (p. 162)
even function (p. 133)
everywhere discontinuous (p. 159)
extremum (p. 171)
horizontal asymptote (p. 180)
horizontal line test (p. 153)
image point (p. 127)
increasing function (p. 164)
infinite discontinuity (p. 159)
inverse function (pp. 152, 155)
inverse process (p. 154)
inversely proportional (p. 191)
inverse relations (p. 152)
joint variation (p. 192)
jump discontinuity (p. 159)
line symmetry (p. 129)
maximum (p. 171)
minimum (p. 171)
monotonicity (p. 163)
odd function (p. 133)
parent graph (p. 137)
point discontinuity (p. 159)
point of inflection (p. 171)
point symmetry (p. 127)
rational function (p. 180)
relative extremum (p. 172)
relative maximum (p. 172)
relative minimum (p. 172)
slant asymptote (p. 183)
symmetry with respect to the origin (p. 128)
vertical asymptote (p. 180)

UNDERSTANDING AND USING THE VOCABULARY

Choose the correct term to best complete each sentence.

1. An (odd, even) function is symmetric with respect to the y-axis.
2. If you can trace the graph of a function without lifting your pencil, then the graph is (continuous, discontinuous).
3. When there is a value in the domain for which a function is undefined, but the pieces of the graph match up, then the function has (infinite, point) discontinuity.
4. A function f is (decreasing, increasing) on an interval I if and only if for every a and b contained in I, $f(a) > f(b)$ whenever $a < b$.
5. When the graph of a function is increasing to the left of $x = c$ and decreasing to the right of $x = c$, then there is a (maximum, minimum) at c.
6. A (greatest integer, rational) function is a quotient of two polynomial functions.
7. Two relations are (direct, inverse) relations if and only if one relation contains the element (b, a) whenever the other relation contains the element (a, b).
8. A function is said to be (monotonic, symmetric) on an interval I if and only if the function is increasing on I or decreasing on I.
9. A (horizontal, slant) asymptote occurs when the degree of the numerator of a rational expression is exactly one greater than that of the denominator.
10. (Inverse, Joint) variation occurs when one quantity varies directly as the product of two or more other quantities.

interNET CONNECTION For additional review and practice for each lesson, visit: **www.amc.glencoe.com**

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 3-1 Use algebraic tests to determine if the graph of a relation is symmetrical.

- Determine whether the graph of $f(x) = 4x - 1$ is symmetric with respect to the origin.

$f(-x) = 4(-x) - 1$
$\quad = -4x - 1$
$-f(x) = -(4x - 1)$
$\quad = -4x + 1$

The graph of $f(x) = 4x - 1$ is not symmetric with respect to the origin because $f(-x) \neq -f(x)$.

REVIEW EXERCISES

Determine whether the graph of each function is symmetric with respect to the origin.

11. $f(x) = -2x$ **12.** $f(x) = x^2 + 2$
13. $f(x) = x^2 - x + 3$ **14.** $f(x) = x^3 - 6x + 1$

Determine whether the graph of each function is symmetric with respect to the *x*-axis, *y*-axis, the line *y* = *x*, the line *y* = −*x*, or none of these.

15. $xy = 4$ **16.** $x + y^2 = 4$
17. $x = -2y$ **18.** $x^2 = \frac{1}{y}$

Lesson 3-2 Identify transformations of simple graphs.

- Describe how the graphs of $f(x) = x^2$ and $g(x) = x^2 - 1$ are related.

Since 1 is subtracted from $f(x)$, the parent function, $g(x)$ is the graph of $f(x)$ translated 1 unit down.

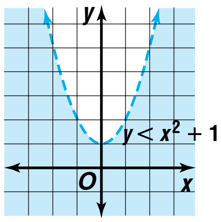
Describe how the graphs of *f*(*x*) and *g*(*x*) are related.

19. $f(x) = x^4$ and $g(x) = x^4 + 5$
20. $f(x) = |x|$ and $g(x) = |x + 2|$
21. $f(x) = x^2$ and $g(x) = 6x^2$
22. $f(x) = [\![x]\!]$ and $g(x) = \left[\!\left[\frac{3}{4}x\right]\!\right] - 4$

Lesson 3-3 Graph polynomial, absolute value, and radical inequalities in two variables.

- Graph $y < x^2 + 1$.

The boundary of the inequality is the graph of $y = x^2 + 1$. Since the boundary is not included, the parabola is dashed.



Graph each inequality.

23. $y > |x + 2|$ **24.** $y \leq -2x^3 + 4$
25. $y < (x + 1)^2 + 2$ **26.** $y \geq \sqrt{2x - 3}$

Solve each inequality.

27. $|4x + 5| > 7$
28. $|x - 3| + 2 \leq 11$

Lesson 3-4 Determine inverses of relations and functions.

- Find the inverse of $f(x) = 4(x - 3)^2$.

$y = 4(x - 3)^2$ *Let y = f(x).*
$x = 4(y - 3)^2$ *Interchange x and y.*
$\frac{x}{4} = (y - 3)^2$ *Solve for y.*

$\pm\sqrt{\frac{x}{4}} = y - 3$

$3 \pm \sqrt{\frac{x}{4}} = y$ So, $f^{-1}(x) = 3 \pm \sqrt{\frac{x}{4}}$.

Graph each function and its inverse.

29. $f(x) = 3x - 1$ **30.** $f(x) = -\frac{1}{4}x + 5$
31. $f(x) = \frac{2}{x} + 3$ **32.** $f(x) = (x + 1)^2 - 4$

Find $f^{-1}(x)$. Then state whether $f^{-1}(x)$ is a function.

33. $f(x) = (x - 2)^3 - 8$
34. $f(x) = 3(x + 7)^4$

OBJECTIVES AND EXAMPLES

Lesson 3-5 Determine whether a function is continuous or discontinuous.

- Determine whether the function $y = \frac{x}{x + 4}$ is continuous at $x = -4$.

  Start with the first condition of the continuity test. The function is not defined at $x = -4$ because substituting -4 for x results in a denominator of zero. So the function is discontinuous at $x = -4$.

REVIEW EXERCISES

Determine whether each function is continuous at the given *x*-value. Justify your response using the continuity test.

35. $y = x^2 + 2$; $x = 2$

36. $y = \frac{x - 3}{x + 1}$; $x = -1$

37. $f(x) = \begin{cases} x + 1 \text{ if } x \leq 1 \\ 2x \text{ if } x > 1 \end{cases}$; $x = 1$

Lesson 3-5 Identify the end behavior of functions.

- Describe the end behavior of $f(x) = 3x^4$.

  Make a chart investigating the value of f(x) for very large and very small values of x.

x	f(x)
−10,000	3×10^{16}
−1000	3×10^{12}
−100	3×10^{8}
0	0
100	3×10^{8}
1000	3×10^{12}
10,000	3×10^{16}

$y \to \infty$ as $x \to \infty$,
$y \to \infty$ as $x \to -\infty$

Describe the end behavior of each function.

38. $y = 1 - x^3$

39. $f(x) = x^9 + x^7 + 4$

40. $y = \frac{1}{x^2} + 1$

41. $y = 12x^5 + x^3 - 3x^2 + 4$

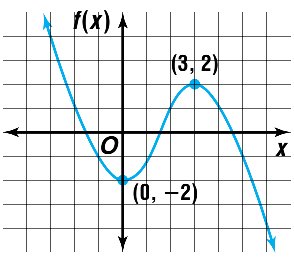
Determine the interval(s) for which the function is increasing and the interval(s) for which the function is decreasing.

42. $y = -2x^3 - 3x^2 + 12x$

43. $f(x) = |x^2 - 9| + 1$

Lesson 3-6 Find the extrema of a function.

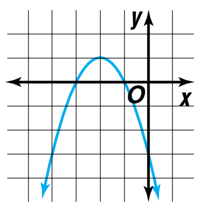
- Locate the extrema for the graph of $y = f(x)$. Name and classify the extrema of the function.



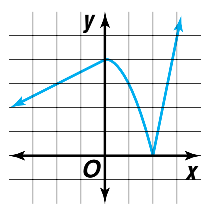
  The function has a relative minimum at $(0, -2)$ and a relative maximum at $(3, 2)$.

Locate the extrema for the graph of $y = f(x)$. Name and classify the extrema of the function.

44.



45.



Determine whether the given critical point is the location of a *maximum*, a *minimum*, or a *point of inflection*.

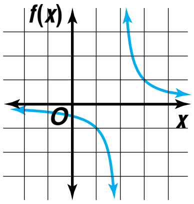
46. $x^3 - 6x^2 + 9x$, $x = 3$

47. $4x^3 + 7$, $x = 0$

OBJECTIVES AND EXAMPLES

Lesson 3-7 Graph rational functions.

- The graph at the right shows a transformation of $f(x) = \frac{1}{x}$. Write an equation of the function.

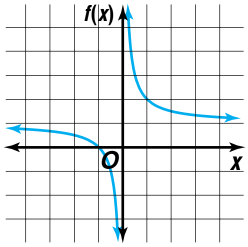


The graph of $f(x) = \frac{1}{x}$ has been translated 2 units to the right. So, the equation of the function is $f(x) = \frac{1}{x-2}$.

REVIEW EXERCISES

Each graph below shows a transformation of $f(x) = \frac{1}{x}$. Write an equation of each function.

48.



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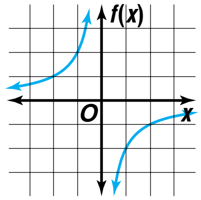

Use the parent graph $f(x) = \frac{1}{x}$ to graph each equation. Describe the transformation(s) that have taken place. Identify the new locations of the asymptotes.

50. $\frac{3}{x+2}$

51. $\frac{2x-5}{x-3}$

OBJECTIVES AND EXAMPLES

Lesson 3-7 Determine vertical, horizontal, and slant asymptotes.

- Determine the equation of the horizontal asymptote of $g(x) = \frac{5x-9}{x-3}$.

To find the horizontal asymptote, use division to rewrite the rational expression as a quotient.

$$\begin{array}{r} 5 \\ x-3\overline{)5x-9} \\ \underline{5x-15} \\ 4 \end{array}$$

Therefore, $g(x) = 5 + \frac{4}{x-3}$. As $x \to \pm\infty$, the value of $\frac{4}{x-3}$ approaches 0. The value of $5 + \frac{4}{x-3}$ approaches 5. The line $y = 5$ is the horizontal asymptote.

REVIEW EXERCISES

Determine the equations of the vertical and horizontal asymptotes, if any, of each function.

52. $f(x) = \frac{x}{x-1}$

53. $g(x) = \frac{x^2+1}{x+2}$

54. $h(x) = \frac{(x-3)^2}{x^2-9}$

55. Does the function $f(x) = \frac{x^2+2x+1}{x}$ have a slant asymptote? If so, find an equation of the slant asymptote. If not, explain.

OBJECTIVES AND EXAMPLES

Lesson 3-8 Solve problems involving direct, inverse, and joint variation.

- If y varies inversely as the square of x and $y = 8$ when $x = 3$, find x when $y = 6$.

$$\frac{x_1^n}{y_2} = \frac{x_2^n}{y_1}$$

$$\frac{3^2}{6} = \frac{x_2}{8} \quad \textit{n = 2, } x_1 = 3,\ y_1 = 8,\ y_2 = 6$$

$6x_2^2 = 72$ *Cross multiply.*

$x_2^2 = 12$ *Divide each side by 6.*

$x_2 = \sqrt{12}$ or $2\sqrt{3}$

REVIEW EXERCISES

Find the constant of variation and use it to write an equation for each statement. Then solve the equation.

56. If y varies jointly as x and z and $y = 5$ when $x = -4$ and $z = -2$, find y when $x = -6$ and $z = -3$.

57. If y varies inversely as the square root of x and $y = 20$ when $x = 49$, find x when $y = 10$.

58. If y varies directly as the square of x and inversely as z and $y = 7.2$ when $x = 0.3$ and $z = 4$, find y when $x = 1$ and $z = 40$.

APPLICATIONS AND PROBLEM SOLVING

59. Manufacturing The length of a part for a bicycle must be 6.5 ± 0.2 centimeters. If the actual length of the part is x, write an absolute value inequality to describe this situation. Then find the range of possible lengths for the part. *(Lesson 3-3)*

60. Consumer Costs A certain copy center charges users \$0.40 for every minute or part of a minute to use their computer scanner. Suppose that you use their scanner for x minutes, where x is any real number greater than 0. *(Lesson 3-4)*

a. Sketch the graph of the function, $C(x)$, that gives the cost of using the scanner for x minutes.

b. What are the domain and range of $C(x)$?

c. Sketch the graph of $C^{-1}(x)$.

d. What are the domain and range of $C^{-1}(x)$?

e. What real-world situation is modeled by $C^{-1}(x)$?

61. Sports One of the most spectacular long jumps ever performed was by Bob Beamon of the United States at the 1968 Olympics.

His jump of 8.9027 meters surpassed the world record at that time by over half a meter! The function $h(t) = 4.6t - 4.9t^2$ describes the height of Beamon's jump (in meters) with respect to time (in seconds). *(Lesson 3-6)*

a. Draw a graph of this function.

b. What was the maximum height of his jump?

ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. Write and then graph an equation that exhibits symmetry with respect to the

   a. x-axis.
   b. y-axis.
   c. line $y = x$.
   d. line $y = -x$.
   e. origin.

2. Write the equation of a parent function, other than the identity or constant function, after it has been translated right 4 units, reflected over the x-axis, expanded vertically by a factor of 2, and translated 1 unit up.

3. A graph has one absolute minimum, one relative minimum, and one relative maximum.

   a. Draw the graph of a function for which this is true.
   b. Name and classify the extrema of the function you graphed.

Additional Assessment See page A58 for Chapter 3 practice test.

Unit 1 interNET Project

TELECOMMUNICATION

Sorry, You are Out of Range for Your Telephone Service . . .

- Research several cellular phone services to determine their initial start-up fee, equipment fee, the monthly service charge, the charge per minute for calls, and any other charges.
- Compare the costs of the cellular phone services you researched by writing and graphing equations.

Determine which cellular phone service would best suit your needs. Write a paragraph to explain your choice. Use graphs and area maps to support your choice.

PORTFOLIO

Choose one of the functions you studied in this chapter. Describe the graph of the function and how it can be used to model a real-life situation.

THE PRINCETON REVIEW

TEST-TAKING TIP

If a problem seems to require lengthy calculations, look for a shortcut. There is probably a quicker way to solve it. Try to eliminate fractions and decimals. Try factoring.

More Algebra Problems

SAT and ACT tests include quadratic expressions and equations. You should be familiar with common factoring formulas, like the difference of two squares or perfect square trinomials.

$a^2 - b^2 = (a + b)(a - b)$
$a^2 + 2ab + b^2 = (a + b)^2$
$a^2 - 2ab + b^2 = (a - b)^2$

Some problems involve systems of equations. Simplify the equations if possible, and then add or subtract them to eliminate one of the variables.

ACT EXAMPLE

1. If $\frac{x^2 - 9}{x + 3} = 12$, then $x = ?$

A 10 **B** 15 **C** 17 **D** 19 **E** 20

HINT Look for factorable quadratics.

Solution Factor the numerator and simplify.

$$\frac{(x + 3)(x - 3)}{x + 3} = 12$$
$$x - 3 = 12 \quad \frac{x+3}{x+3} = 1$$
$$x = 15 \quad \textit{Add 3 to each side.}$$

The answer is choice **B.**

Alternate Solution You can also solve this type of problem, with a variable in the question and numbers in the answer choices, with a strategy called "backsolving."

Substitute each answer choice for the variable into the given expression or equation and find which one makes the statement true.

Try choice A. For $x = 10$, $\frac{10^2 - 9}{10 + 3} \neq 12$.

Try choice B. For $x = 15$, $\frac{15^2 - 9}{15 + 3} = 12$.

Therefore, choice **B** is correct.

If the number choices were large, then calculations, even using a calculator, would probably take longer than solving the problem using algebra. In this case, it is *not* a good idea to use the backsolving strategy.

SAT EXAMPLE

2. If $x = y + 1$ and $y \geq 1$, then which of the following must be equal to $x^2 - y^2$?

A $(x - y)^2$ **B** $x^2 - y - 1$ **C** $x + y$
D $x^2 - 1$ **E** $y^2 + 1$

HINT This difficult problem has variables in the answers. It can be solved by using algebra or the "Plug-in" strategy.

Solution Notice the word *must.* This means the relationship is true for all possible values of x and y.

To use the Plug-In strategy, choose a number greater than 1 for y, say 4. Then x must be 5. Since $x^2 - y^2 = 25 - 16$ or 9, check each expression choice to see if it is equal to 9 when $x = 5$ and $y = 4$.

Choice A: $(x - y)^2 = 1$
Choice B: $x^2 - y - 1 = 20$
Choice C: $x + y = 9$

Choice **C** is correct.

Alternate Solution You can also use algebraic substitution to find the answer. Recall that $x^2 - y^2 = (x + y)(x - y)$. The given equation, $x = y + 1$, includes y. Substitute $y + 1$ for x in the second term.

$$\begin{aligned} x^2 - y^2 &= (x + y)(x - y) \\ &= (x + y)[(y + 1) - y] \\ &= (x + y)(1) \\ &= x + y \end{aligned}$$

This is choice **C.**

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

1. For all $y \neq 3$, $\dfrac{y^2 - 9}{3y - 9} = ?$

   A y B $\dfrac{y+1}{8}$

   C $y + 1$ D $\dfrac{y}{3}$

   E $\dfrac{y+3}{3}$

2. If $x + y = z$ and $x = y$, then all of the following are true EXCEPT

   A $2x + 2y = 2z$

   B $x - y = 0$

   C $x - z = y - z$

   D $x = \dfrac{z}{2}$

   E $z - y = 2x$

3. The Kims drove 450 miles in each direction to Grandmother's house and back again. If their car gets 25 miles per gallon and their cost for gasoline was \$1.25 per gallon for the trip to Grandmother's house, but \$1.50 per gallon for the return trip, how much *more* money did they spend for gasoline returning from Grandmother's house than they spent going to Grandmother's?

   A \$2.25 B \$4.50

   C \$6.25 D \$9.00

   E \$27.00

4. If $x + 2y = 8$ and $\dfrac{x}{2} - y = 10$, then $x = ?$

   A -7 B 0

   C 10 D 14

   E 28

5. $\dfrac{900}{10} + \dfrac{90}{100} + \dfrac{9}{1000} =$

   A 90.09 B 90.099

   C 90.909 D 99.09

   E 999

6. For all x, $(10x^4 - x^2 + 2x - 8) - (3x^4 + 3x^3 + 2x + 9) = ?$

   A $7x^4 - 3x^3 - x^2 - 17$

   B $7x^4 - 4x^2 - 17$

   C $7x^4 + 3x^3 - x^2 + 4x$

   D $7x^4 + 2x^2 + 4x$

   E $13x^4 - 3x^3 + x^2 + 4x$

7. If $\dfrac{n}{8}$ has a remainder of 5, then which of the following has a remainder of 7?

   A $\dfrac{n+1}{8}$ B $\dfrac{n+2}{8}$

   C $\dfrac{n+3}{8}$ D $\dfrac{n+5}{8}$

   E $\dfrac{n+7}{8}$

8. If $x > 0$, then $\dfrac{\sqrt{100x^2 + 600x + 900}}{x + 3} = ?$

   A 9 B 10 C 30 D 40

   E It cannot be determined from the information given.

9. What is the value of c?

   Given: $a + b = c$
   $a - c = 5$
   $b - c = 3$

   A -10
   B -8
   C -5
   D -3
   E 3

10. **Grid-In** If $4x + 2y = 24$ and $\dfrac{7y}{2x} = 7$, then $x = ?$

interNET CONNECTION **SAT/ACT Practice** For additional test practice questions, visit: **www.amc.glencoe.com**